Section I
Radiative Transfer Theory

V.V. Sobolev and Analytical Radiative Transfer Theory

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The review of Sobolev's publications on the analytical radiative transfer theory is presented. A short review is also given of the results published by his disciples.

1 Introduction

The basic equations of the Radiative Transfer Theory (RTT) were formulated at the turn of the 20th century. Initially transfer theory developed as a purely analytical instrument since the calculation of radiation fields was problematic with the computational facilities of that time. Between 1940 and 1980 the exact and sufficiently accurate approximate solutions to the basic equations of the theory were found, and for various limiting cases the asymptotic theory was developed. The peculiarities and difficulties of the description and computation of multiple scattering were thus revealed. By comparing numerical results with the analytical solutions it became possible to evaluate the benefits and drawbacks of various numerical methods and give estimates of their accuracy.

V.V. Sobolev made a definitive contribution to the creation of the analytical RTT. In the 1940s he developed a method of calculating populations of atomic levels in expanding non-planar dilute gaseous media, the method which is still in use. This method is known now as *the Sobolev theory.* He also developed an effective approximate method to solve problems of anisotroping multiple light scattering. As early as in 1941, he formulated the approximation of Complete Frequency Redistribution (CFR) in problems of radiative transfer in spectral lines. In the 1950s he developed the method of exact solution of the basic integral equations describing multiple light scattering, both monochromatic and with CFR. He was also the first to investigate multiple scattering of polarized radiation and non-stationary radiation fields. He applied his theoretical findings to the interpretation of observations of many types of astrophysical objects. Dozens of former Sobolev's students form a team of theorists known as the Sobolev astrophysical school. In what follows we present a brief review of the main results found by V.V. Sobolev and his disciples.

We begin with the description of contributions to the analytical RTT by V.A. Ambartsumian (who was Sobolev's Ph.D. adviser).

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2 Contribution of V.A. Ambartsumian

V.A. Ambartsumian founded the chair of astrophysics (1934) in the St. Petersburg (Leningrad) University. He published the first Russian manual on theoretical astrophysics [1].

He studied the radiation regime in an infinite plane medium with sources at the infinite depth, thus modeling deep layers of a semi-infinite medium with anisotropic monochromatic scattering [2].

Ambartsumian revealed the important role of radiation pressure by spectral line photons in the dynamics of planetary nebulae and stellar envelopes, particularly, the pressure exerted by the photons of the hydrogen L_α-line [3]. He suggested a new method to describe the influence of absorption lines on the temperature regime in stellar atmospheres [4].

Ambartsumian introduced innovative approaches to RTT problems known as the invariance principles and the method of adding of layers [5]. Using these new methods he expressed the reflection and transmission coefficients of a plane layer which are functions of two angular variables in terms of auxiliary functions of one variable [6]. For these functions he found nonlinear integral equations and studied the asymptotic behavior of their solutions for the case of a layer of large optical thickness [7, 8].

He expressed the mean number and the mean square of the number of scattering events in terms of the radiation intensity [9].

He studied also the problem of light scattering in semi-infinite medium with reflecting surface [10].

The main Ambartsumian's publications on RTT are reprinted in the book [11]. The proceedings of the conference dedicated to the 40th anniversary of the Invariance Principle are published in [12].

V.A. Ambartsumian studied many other astrophysical problems: the lifetimes of stars, star clusters, stellar associations, the Milky Way brightness fluctuations, formation of galaxies, variable stars, etc.

V.V. Sobolev continued studies of his teacher in RTT. He discovered new branches and created new methods of RTT, formulated and solved a lot of new problems.

3 Early Sobolev's publications

V.V. Sobolev proposed a method of approximate solution to the problem of anisotropic scattering of monochromatic radiation. According to this method the first scattering is taken into account exactly, with the real phase function, whereas higher order scatterings are treated approximately, with the two-term phase function [13]. V.V. Sobolev applied the developed theory to terrestrial and planetary atmospheres [14, 15]. Later, this approximate method was applied to problems with spherical geometry: scattering in a homogeneous sphere with a point source at its center [16] (a model of dust nebula) and in a spherical shell illuminated by a parallel radiation flux [17] (model of a planetary atmosphere, coauthor I.N. Minin).

V.V. Sobolev showed that the idea of accelerated expansion of planetary nebulae adopted at that time is incorrect because it was based on the assumption that line radiation does not change its frequency when scattered. In fact, the scattered photon reduce its frequency because a part of its momentum and energy passes to the scattering atom. Therefore the full momentum of stellar radiation is not transmitted to the nebula matter and does not accelerate it: radiation simply leaves the nebula in the wings of the line [18]. In [19] V.V. Sobolev simplified the calculation of the radiation regime in infinite plane medium.

The problems of radiative transfer in expanding media were studied in Sobolev's doctoral thesis and in his famous book [20]. The equations determining the populations of atomic levels were derived and solved using the method of local scattering. The method is known as *the Sobolev approximation* and is widely used till now. The essence of the method is the following. If a medium expands with a velocity gradient, the radiation in a line ceases to interact with atoms when it propagates in places where gas velocity is substantially different from the velocity at the site of its emission. As a result, the line radiation is not reabsorbed and propagates freely. The scattering becomes local. Due to this effect, in media moving with large gradient of gas velocity atomic excitation and degree of ionization change drastically [21].

Later, for the special case of the two-level atom and a constant velocity gradient in plane media, the integral equation was formulated, with the kernel depending on the absolute value of the difference of the arguments. The approximate solution of the equation was found using "on the spot" approximation [22].

In two papers [23] and [24] (with V.V. Ivanov) the intensities of hydrogen lines and the Balmer decrement in the spectra of hot stars were calculated. Lines are formed in their envelopes. By applying the approximation of local scattering, the equations governing the populations of atomic levels were reduced to algebraic ones.

4 Monochromatic scattering

4.1 Polarized radiation and non-stationary radiation fields

In [25] V.V. Sobolev formulated transfer equations for linearly polarized radiation for the case of Rayleigh scattering. He found the behavior of two intensities and the corresponding source functions in deep layers of semi-infinite medium. He also found the degree of polarization of the radiation emerging from purely electronic semi-infinite medium with the sources at infinite depth, thus modeling a hot atmosphere of an early type star. The largest degree of polarization, 11.7%, is reached at the limb of the stellar disk. This is known as the *Sobolev–Chandrasekhar polarization limit.*

Later V.V. Sobolev published several papers on Rayleigh scattering (with V.M. Loskutov). They calculated fields of polarized radiation in plane slabs for several distributions of primary sources [26]. The results were used for the interpretation of observed polarization of X-ray sources [27] and quasars [28].

For studying non-stationary radiation fields in stationary media V.V. Sobolev introduced two characteristic times [29]: t_1, the mean time a photon spends while absorbed by an atom and t_2, the mean time between two consecutive scatterings of a photon. He derived the equations describing non-stationary radiation fields in one-dimensional approximation and solved them for the case $t_2 = 0$, both for final and infinite optical thickness of the medium. The solutions thus found were used to interpret peculiarities of radiation fields in the ejecta of novae (see [30]).

Later on V.V. Sobolev continued studying the non-stationary scattering with his coauthor A.K. Kolesov [31, 32]. They presented the formulas and numerical data for the solutions to the problem of a point source in an infinite and semi-infinite one-dimensional media for alternative cases $t_2 \ll t_1$ and $t_1 \ll t_2$. The results were applied to interpret the flares of UV Ceti stars.

4.2 Reflecting boundaries and inhomogeneous media

V.V. Sobolev derived the equations for radiation fields in a plane media with a reflecting lower surface. Two particular cases were considered in more detail: orthotropic and mirror reflection. In the former case the all quantities with the reflecting surface were expressed by simple relations in terms that without it [33]. The results for the case of a mirror boundary were published in the book [30] and applied to the scattering in a cloudy slab of large optical thickness above the surface of the sea.

The problem of scattering in plane media if the probability of photon survival λ depends on the depth τ was considered [34, 35]. The calculations of the albedo and brightness coefficients were made for the cases:

1) λ is piecewise constant;
2) λ is an exponent of optical depth $\lambda = \lambda_0 \, e^{-m\tau}$ or the sum of such exponents;
3) λ is a superposition (integral) of exponents.

Later the degree of polarization of the radiation emergent from the semi-infinite medium was calculated for the case 2) [36] (coauthor V.M. Loskutov).

4.3 New methods of calculation of radiation fields

V.V. Sobolev formulated the concept of photon escape probability from a medium: the product $2\pi p\,(\tau, \eta)\,\mathrm{d}\eta$ denotes the probability for a photon absorbed at a depth τ in isotropically scattering semi-infinite atmosphere to escape from this medium at an angle $\arccos\eta$ within a solid angle of $2\pi\,\mathrm{d}\eta$ after an arbitrary number of scatterings. It is easy to obtain the equations and relations for the escape probability from simple considerations. If this function is found, it is possible with the known power of primary sources to calculate the intensity of emergent radiation by direct integration [30]. Apart from this, the majority of the functions and equations of RTT got the probability interpretation. The concept of the escape probability was applied to many problems of RTT for deducing the equations and solving them.

Another method which was applied by V.V. Sobolev is transformation from equations with integrals on optical depth τ to linear equations with integrals on angular variables which is equivalent to application of the Laplace transform. Such equations were derived for brightness coefficients, functions of one variable in terms of which these coefficients were expressed and other functions. It was such type equations that were used for the calculations of polarization fields with the Rayleigh scattering and were mentioned above.

4.4 Asymptotic theory of monochromatic scattering

The complete asymptotics of the source function and of the intensity in deep layers of a semi-infinite medium for the reflection problem were obtained by V.V. Sobolev (see [37]) using the relations between characteristics of anisotropic scattering in infinite and semi-infinite media which were found with the summation of layers method. Using these results and with the same method V.V. Sobolev deduced asymptotics for the brightness coefficients and other functions when the optical thickness of a slab τ_0 was large [38].

Another domain for which the asymptotic formulas were found is a nearly pure scattering when the survival probability is very close to unity: $1-\lambda \ll 1$. Expansions of various functions on the power of $\sqrt{1-\lambda}$ (the first or second) were obtained. The results are given in the book [39].

5 Scattering in lines and the resolvent method

5.1 Frequency redistribution

V.V. Sobolev directed essential efforts to the study of scattering in spectral lines.

The laws describing the transformation of photon frequency in single scattering were deduced but they were too complicate and did not allow to solve the problem of line formation. Several authors (T. Holstein, L.M. Biberman, V.V. Sobolev, and others) proposed the approximation of complete redistribution in frequency (CFR), which implied that the photon frequencies before and after scattering do not correlate. In other words, the absorption and emission coefficients depend on frequency equally. The following additional approximations were accepted: atoms of the same kind have only two discrete levels (the two-level approximation) and continuum constant within the line; both radiative and collisional transitions are possible between these levels; the induced radiation was not taken into account because it leads to nonlinear equations, which do not permit analytical investigation.

To begin V.V. Sobolev derived some frequency redistribution laws and accepted as the approximation the CFR. Then he considered scattering in a one-dimensional medium and obtained differential equations for the intensity and integral equation for the source function. He solved them for homogeneous distribution of the sources and found the emissivity, the density and flux of energy, the light pressure and the emission profiles of emergent radiation. The results were close for various redistribution laws and strongly differed from the monochromatic scattering.

Then various equations were obtained for the probability $p(\tau, x_1, x_2)$ of photon escape from the medium of optical thickness τ_0 in line from the depth τ (x_1 and x_2 are dimensionless frequencies of the emitted and escaping photons). Also the equations were derived for the two introduced functions $\varphi(x, \tau_0)$ and $\psi(x, \tau_0)$. After that V.V. Sobolev found the equation for the brightness coefficients. The equations were solved for CFR and the profiles of the forming absorption lines were calculated. Better agreement with the observable ones than for monochromatic scattering were achieved. All these results are in his book [30].

The integro-differential and integral equations, describing the process of multiple photon scattering in spectral line in a plane layer on the assumption of CFR, were derived by V.V. Sobolev in [40]. The approximate solution of the integral equation based on the principle of local scattering was found. It is usually known as on the spot approximation. Later V.V. Sobolev developed the exact theory of multiple scattering known as resolvent method. At first it was done for isotropic monochromatic scattering [41, 42] and then for scattering in line with CFR [43, 44].

5.2 Resolvent method

This method is applicable to equations of the following form:

$$S(\tau) = S_0(\tau) + \frac{\lambda}{2} \int\limits_{\tau_*}^{\tau_0} K(|\tau - \tau'|) S(\tau')\, d\tau'. \tag{1}$$

This equation is the basic integral equation of RTT. Here $S_0(\tau)$ is a given, and $S(\tau)$ is the sought-for source function, λ is photon survival probability per scattering. The limits of integration τ_* and τ_0 are the "depths" of the lower and upper boundaries of a plane medium. If $-\tau_* = \tau_0 = \infty$, the medium is infinite; if $\tau_* > -\infty$ and $\tau_0 = \infty$, the medium is semi-infinite; if $\tau_0 < \infty$, it is a finite plane slab. In the last two cases it can be assumed that $\tau_* = 0$. The kernel function $K(\tau)$ for both monochromatic and CFR scattering can be represented as a superposition (integral) of exponentials.

The resolvent is defined as a function that allows one to find the solution of Eq. (1) for arbitrary given $S_0(\tau)$

$$S(\tau) = S_0(\tau) + \int\limits_{\tau_*}^{\tau_0} \Gamma(\tau, \tau') S_0(\tau')\, \mathrm{d}\tau'. \tag{2}$$

The notations of resolvents are as follows: for an infinite medium it is $\Gamma_\infty(\tau,\tau_1)$, for a semi-infinite medium $\Gamma(\tau,\tau_1)=\Gamma(\tau,\tau_1,\infty)$, and for a finite slab $\Gamma(\tau,\tau_1,\tau_0)$.

For an infinite medium the following obvious relation holds: $\Gamma_\infty(\tau,\tau_1) = \Gamma_\infty(|\tau-\tau_1|,0) \equiv \Phi_\infty(|\tau-\tau_1|)$. V.V. Sobolev has shown that the resolvent of the equation (1) can be expressed in terms of a function of one variable, namely, the particular value of the resolvent with one of its arguments set equal to 0. This function is called the *resolvent function*: $\Phi(\tau,\tau_0)\equiv\Gamma(\tau,0,\tau_0)$. If $\tau_*=0$, the explicit expression of Γ in terms of Φ is rather complicated

$$\Gamma(\tau,\tau_1,\tau_0) = \Phi(|\tau-\tau_1|,\tau_0) + \int\limits_0^{\min(\tau,\tau_1)} \big[\Phi(\tau-t,\tau_0)\Phi(\tau_1-t,\tau_0) - \Phi(\tau_0-\tau+t,\tau_0)\Phi(\tau_0-\tau_1+t,\tau_0)\big]\,dt. \tag{3}$$

For semi-infinite medium one has to set $\tau_0=\infty$ and $\Phi(\infty,\infty)=0$.

For the kernel functions representable as a superposition of exponentials V.V. Sobolev derived linear and nonlinear equations for the Laplace transforms of the resolvents and the resolvent functions as well as equations for resolvent functions themselves of type (1) (with $S_0(\tau) = (\lambda/2)\,K(\tau)$) and of Volterra type. Some of these equations are generalizations of Ambartsumian's equations. For a finite slab alternative equations were derived, with the derivatives with respect to τ_0.

For isotropic monochromatic scattering V.V. Sobolev found the asymptotic form of $\Phi(\tau,\tau_0)$ for $\tau\gg1$[45]. It is expressed in terms of the resolvent function of semi-infinite medium. For the latter the exact explicit expression is known.

V.V. Sobolev encouraged his pupils for further development of the theory of line formation with CFR.

5.3 Inhomogeneous, infinite and spherical media

The scattering theory in inhomogeneous media was extended to the scattering in a spectral line with CFR [46] taking into account continuous absorption. In [47] V.V. Sobolev and E.G. Yanovitsky applied the resolvent method to the case of scattering with variable $\lambda(\tau)$. In [48] the results for the variable λ were summarized.

In [49] it was shown that three problems of monochromatic isotropic scattering in three media, namely: in a semi-infinite medium with an ideally reflecting mirror boundary; in a stationary spherical shell geometrically thin but optically thick with the central source and also in an infinite medium with a point source, are reduced to scattering in infinite medium.

In [50] the case of a smoothly reflecting boundary was reduced to two integral equations and with two resolvent functions. In [51] along with the fact that the smooth reflection from the boundary is not ideal, the changing direction of radiation when crossing it was taken into account because the refraction indexes differ on its two sides. For the problem of diffuse reflection the equations were obtained for the azimuthal harmonics of the reflection coefficient by assuming that such harmonics calculated without reflection from the boundary are known. Also, the equations were deduced for characteristics of emergent radiation in the Milne problem and of the regime of the radiation field in deep layers. These problems are to model the light scattering at sea.

Several papers were devoted to monochromatic scattering in a homogeneous sphere and in a spherical envelope. In [52] the problem of scattering in the sphere with spherically symmetric sources was reduced to the problem of a plane slab of double optical thickness. In [53] the asymptotic formulas were obtained for the intensity of emergent radiation $I(\eta, \tau_0)$ when there is a point source in the center of the sphere or in the center of a thin spherical envelope and when the optical thicknesses of the sphere and the envelope τ_0 are large. V.V. Sobolev and A.K. Kolesov found more exact asymptotics of $I(\eta, \tau_0)$ for illuminating a sphere both by a radiation flux [54] and by a point source in the center [55]. The summary of these researches was presented in [56].

5.4 The resolvent method for anisotropic scattering

The equations governing anisotropic scattering with an arbitrary phase function contain integrals over three variables τ, η and ϕ. It is possible to expand characteristics of scattering in the Fourier series (or finite sum) on azimuth ϕ, to separate azimuth harmonics and to deduce separate equations with double integrals for each of the harmonics.

The intensity of the emergent radiation in the problem of reflection and transmission may be expressed in terms of functions $\varphi_i^m(\eta, \tau_0)$ and $\psi_i^m(\eta, \tau_0)$. If the number of terms in the expansion of the phase function in the Legendre polynomials equals $n+1$, to get the harmonic with number m $(i = m, m+1, \ldots, n)$ one has to find $2(n - m + 1)$ such functions. For semi-infinite medium the functions $\psi_i^m(\eta, \infty) = 0$. For semi-infinite medium V.V. Sobolev expressed all the functions $\varphi_i^m(\eta)$ for each of the harmonics in terms of one function $H^m(\eta)$. The functions $\varphi_i^m(\eta, \tau_0)$ and $\psi_i^m(\eta, \tau_0)$ were expressed in terms of two functions, $X^m(\eta, \tau_0)$ and $Y^m(\eta, \tau_0)$. The polynomials depending on η and λ entered these expressions as factors. They are given by recurrent relations. The resolvent for each of the harmonics is expressed in terms of one resolvent function $\Phi_m(\tau, \tau_0)$. These results are summarized in Sobolev's book [39].

6 Other Sobolev's works on RTT

6.1 Number of scatterings; strongly peaked phase function

In four of V.V. Sobolev's papers [57] the numbers of scatterings were expressed through the functions introduced in other works. In the fourth paper for the case of scattering in the spectral line with CFR in finite slab were obtained sufficiently narrow upper and lower estimations of the number of scatterings for large optical thickness.

In the case of strongly elongated forward phase function V.V. Sobolev expanded the intensity according the Taylor formula of the second order in the powers of the difference between the polar angles of the scattered and the incident radiation, and replaced the integral over angles with the differential operator. With the help of the obtained equation he found the light regime in deep layers [58].

6.2 Scattering in planetary atmospheres

Twenty years after publishing [14] V.V. Sobolev resumed the study of scattering characteristics in the Venus atmosphere. In the first paper [59] he calculated the reflection coefficient for a two-term phase function with the terms proportional to $\sqrt{1-\lambda}$. In the second paper two models of the atmosphere were adopted: the homogeneous one consisting of molecules and large-grained particles; and the two-layer one in which a molecular slab is placed above a cloudy slab. The dependencies of the degree of polarization on the phase and wavelength were found.

In two papers [60, 61] (coauthors I.N. Minin) the radiation of planetary atmosphere was described for isotropic scattering. The atmosphere was assumed to be plane (with the dependence $\lambda(\tau)$), but the incident angles of solar radiation on the plane were chosen to be the same as those of a parallel flux on spherical atmosphere. The effect of orthotropic reflection from the surface was taken into account.

In [62] the formulas for the profile r_ν and equivalent width W of a line in a certain place of the planetary disk and from the whole disk were derived as functions of phase. In [63] a two-layer atmosphere was constructed of a semi-infinite medium and an optically thin slab above it with different optical properties (i.e. their phase functions and photon survival probabilities differed). The same formulas were obtained.

6.3 Emission of supernovae and electron scattering

In three papers [64, 65, 66] V.V. Sobolev (in the third coauthor A.K. Kolesov) calculated the continuous spectra, light curves, optical thickness of envelopes and spectrophotometric temperatures of supernovae on the early stages of expansion of the ejecta. It was adopted that the radiation of envelope was under strong effect of electron scattering.

The effect of electron scattering on the spectra of stationary hot stars was studied in [67, 68], which continued the study in [35]. The emergent flux and specrophotometric temperature were calculated using as a tool linear integral equations with the integrals on angular variable.

6.4 Global absorption and emission

V.V. Sobolev devoted several papers specially to determination of relation between two parts of radiation energy that enters into a scattering and absorbing medium. One part of this energy undergoes true absorption and transfers to other types of energy. The other part abandons the medium. In the most general form the problem was considered in [69]. The law of redistribution in frequency and direction as the fraction of reemitted photons could be different in different points of the medium of arbitrary form (in [70] scattering was supposed to be isotropic). The amount of energy absorbed from the flux illuminating the medium was shown to be connected with the amount of irradiated energy if the distribution of internal sources was uniform. Analogous relations were obtained for Rayleigh scattering of polarized radiation [71].

The usefulness of the obtained relation was demonstrated for isotropic monochromatic scattering as well as for scattering in a line with CFR in a semi-infinite medium, in a plane slab and in a homogeneous sphere [72]. V.V. Sobolev deduced the integral relations for the intensities of internal and emerging from a plane slab radiation in [73].

7 Contributions of Sobolev's disciples

Here we present a list of the main achievements made by Sobolev's students and disciples. More detailed reviews of their works and the works of other authors are presented in the symposiums proceedings [12, 74, 75, 76].

7.1 I.N. Minin

The papers published with V.V. Sobolev as a coauthor are [17, 60, 61].

I.N. Minin deduced the equation and proposed the method to calculate the radiation transfer in a medium with refraction [77, 78] and obtained the exact expression for the resolvent function for monochromatic isotropic scattering in semi-infinite medium [79].

I.N. Minin used the Laplace transforms on time for solving the non-stationary radiation transfer in a medium with monochromatic scattering and studied it in detail [80, 81]. He showed that in three particular cases when $t_2 = 0$, $t_1 = 0$ and $t_1 = t_2$ the solutions for $\lambda < 1$ can be expressed through the solutions for $\lambda = 1$. The exact and asymptotic formulas for characteristics of the radiation emerging from a finite one-dimensional medium were obtained in three mentioned cases [82].

In [83] many characteristics of radiation field in a semi-infinite medium with arbitrary values of t_1 and t_2 were expressed in terms of one function. The equation determining this function was derived. Time-dependent problems were also solved for non-stationary one-dimensional ($\tau(t) = \tau(0)e^{-\alpha t}$) [84] and inhomogeneous [85] media.

Anisotropic scattering in semi-infinite medium [86] and in the layer of finite optical thickness [87] was investigated. If the number of terms in the expansion of phase function in the Legendre polynomials is equal to $n+1$, then for azimuthal harmonic number m I.N. Minin introduced $(n+1-m)^2$ resolvent functions and derived equations for them. His results are in his reviews [88, 89] and book [90].

7.2 V.V. Ivanov

The derivation of various asymptotics that characterize scattering in a spectral line with CFR directly from the equations [91, 92].

The wide use of the concept of thermalization length $\tau_{\rm t}$ (depending on λ). Its value separates two regions. In the first one (depths $\tau < \tau_{\rm t}$) the scattering in line can be considered as conservative while in the second one ($\tau > \tau_{\rm t}$) photons are thermalized, i.e. the source function becomes proportional to the Planck function [93]. Asymptotic formulas for the resolvent functions $\Phi_\infty(\tau)$ and $\Phi(\tau)$ for $\tau \gg 1$ depend not τ and λ separately but in the essential parts only on $\tau/\tau_{\rm t}$.

Time-variations of the degree of excitation for two-level atoms and of the line profile formed in an infinite homogeneous medium with CFR and $t_1 = 0$ if initially the atoms are completely excited [94]. The leading terms of the asymptotics at large time intervals coincide with those obtained for a more exact law of redistribution.

The detailed description of the asymptotic theory of conservative scattering [95, 96] was made for the Milne problem with isotropic monochromatic scattering as well as for CFR scattering with the absorption coefficient decreasing in the line wings as a power of frequency. The asymptotics of X and Y-functions were expressed in terms of the Bessel functions. For the Doppler profile it was performed earlier [97]. It was the very first result of what is now known as the large-scale description.

The concept of the "mean length of a photon path" $\overline{T}$, i.e. path from the place of photon emission to the place where it is finally absorbed (i.e. thermalized) was introduced and the formulas determining $\overline{T}$ were derived [98].

The formulation of a high accuracy approximate solution [99] to the basic CFR integral equation of RTT, both for half-space and for plane layer of finite thickness.

The description of multiple scattering of spectral line photons as a stochastic process of the Lévy random walks was given. It was used to obtain various asymptotics of CFR RRT (with Sh.A. Sabashvili) [100].

The solution was found to the problem of diffuse reflection and transmission of radiation if $t_1 = 0$, $t_2 = 1$ and the boundary of anisotropically scattering layer of finite optical thickness τ_0 is illuminated by an instant light flash [101] (with S.D. Gutshabash). The asymptotic behavior of the brightness wave escaping the layer was found assuming the thickness of the layer $\tau_0 \gg 1$.

The process of frequency relaxation to CFR due to multiple scatterings with non-CFR redistribution functions was studied [102] (with A.B. Schneeweis).

Generalizations of the invariance principles for a semi-infinite medium with scalar anisotropic scattering [103] and for scattering of polarized radiation were formulated [104] (with H. Domke). The asymptotic forms of the basic functions were found explicitly.

New concepts for treating analytically the so called blanketing effect were introduced: the "partial intensity", i.e. the contribution to the intensity by photons classified both by the value of the absorption coefficient and by the frequency, and the so-called "gray in the average" atmosphere, in which the opacity probability distribution function (OPDF) is the same along the whole spectrum. The equations describing the radiation transfer in such an atmosphere were given [105] (with A.G. Kheinlo).

Molecular and Rayleigh scattering of polarized radiation was studied in detail using the concept of matrix transfer equation (with V.M. Loskutov, S.I. Grachev, and T. Viik). In particular, the so called $\sqrt{\varepsilon}$ law of the scalar theory was generalized to incorporate polarization [106, 107, 108].

V.V. Ivanov with coauthors investigated scattering polarization of radiation in resonance lines under the assumption that angular and CFR frequency redistributions are not correlated [109].

The albedo shifting method when the kernel of the integral equation is changed to another one in order to accelerate the convergence of iterations was developed (with coauthors) [110, 111, 112].

Ivanov's results are published also in books [113, 114] and in review [115].

7.3 A.K. Kolesov

Articles with V.V. Sobolev [31, 32, 54, 55, 66].

In three papers [116, 117, 118] calculations were made for the Henyey–Greenstein phase function.

In the series of papers [119, 120, 121, 122] the radiation fields in two-layer and multilayer media with anisotropic scattering in the layers were studied. In the most general case the layers differed in the values of λ, phase functions and refraction indexes.

The expansions in the elementary solutions of the radiation transfer equation (the Case method) were applied for radiation fields in non-plane media with anisotropic scattering. In [123] the problem of scattering in a homogeneous sphere was reduced to the plane one. The same procedure was made for a point source in an infinite medium [124]. In [125] the expression for the Green function of the point source and in [126] the asymptotics of this function were obtained, and in [127] the intensity of radiation far from the point source in the infinite medium was expanded in the reversed powers of τ. The case of small true absorption was considered separately. In [128] and [129] the Case modes were found and the relations of their orthogonality were formulated for spherical and cylindrical symmetries.

Review [130].

7.4 E.G. Yanovitskij

With V.V. Sobolev [47].

Detailed investigation of anisotropic scattering in inhomogeneous [131] and multilayer media [132, 133, 134] (the first and the third papers with Zh.M. Dlugach).

Some formulas for the pure scattering were shown to coincide for an arbitrary scattering phase function [135].

For a semi-infinite medium [136] and a plane layer [137] the equations were formulated, which have the form of the transfer equation in the so-called pseudo problems of anisotropic scattering. These equations determine the intensity of radiation, which would correspond to the source functions equal to the resolvent functions $\Phi^m(\tau)$ and $\Phi^m(\tau, \tau_0)$ that were introduced by V.V. Sobolev.

A new form of the radiation transfer equation (called Q-form) was deduced, where the intensity was represented as the derivative on the optical depth of some linear integral operator of the same intensity [138].

The results of Yanovitskij and his coauthors can be found in his book [139].

7.5 D.I. Nagirner

Using the methods of the theory of complex variables the exact explicit solutions and their asymptotic forms were obtained for stationary [140, 141] and non-stationary ($t_2 = 0$) [142] multiple scattering with CFR in infinite and semi-infinite media. Large-scale and uniform asymptotics of the resolvent and other functions describing scattering in a plane layer [143] and sphere [144] of large optical thickness and radius were found, in particular, the mean number of scatterings and dispersion.

The exact and asymptotic formulas describing the process of damping of the atomic excitation in a homogeneous infinite medium, the excitation being created instantly at the initial moment [145]. The scattering in a line with CFR and the Lorentz absorption profile with an arbitrary ratio of t_1 and t_2 parameters was assumed.

The method was proposed to calculate the eigenvalues and eigenfunctions of the basic integral equation (continuous $\lambda(u) = 1/V(u)$ for a semi-infinite medium and discrete $\lambda_n(\tau_0) = 1/V(u_n(\tau_0))$ for a layer of finite thickness). The asymptotics (on n and $\tau_0 \gg 1$) of the eigenvalues were found for an optically thick layer [146].

The resolvent function, its asymptotic behavior and the asymptotics of the spectrum of the basic integral equation for a cylinder were obtained [147].

The method to calculate the scattering in a plane layer of finite optical thickness was proposed [148].

Reviews [149, 150, 151, 152, 153] and books [154, 155].

7.6 V.M. Loskutov

With V.V. Sobolev [26, 27, 28, 36] and V.V. Ivanov [108, 109].

With a given value of the characteristic number k the value of λ is found by the expansion in a chain fraction for the Henyey–Greenstein phase function [156].

The full phase matrix for the Rayleigh scattering is represented by the product $\mathbf{A}(\eta,\phi)\mathbf{A}^{\mathrm{T}}(\eta',\phi')$, where $\mathbf{A}(\eta,\phi)$ is a matrix of the size 3×6. This representation separates the angular variables of the incident (η',ϕ') and scattered (η,ϕ) radiation. For the six-term vector of the source functions the system of integral equations was obtained. Within the same approximation, in which redistribution depends on frequency and on angles independently as in [109], the matrix equation for the basic matrix was derived. The polarization degree of reflected radiation was calculated [157].

For the Lorentz absorption profile the polarization characteristics of radiation in a line emergent from a semi-infinite medium were calculated. It was noted that the polarization degree as a function of the absorption profile value (rather than the frequency) depends only slightly on the type of this profile (Lorentz, Doppler or rectangular) [158] (with V.V. Ivanov).

Review [159].

7.7 V.P. Grinin

The non-stationary radiation fields in a semi-infinite medium with anisotropic scattering and illuminated by a parallel external flux or a point source were studied [160]. The full radiation and the dependence of the radiation density on the distance from the source were found. The solutions were expressed through the function introduced by I.N. Minin [83].

The methods to calculate the radiation fields in expanding media were proposed [161].

The concept of non-local (large-scale) radiative interaction was introduced and the equations describing the interaction were deduced [162] (coauthor S.I. Grachev) and [163].

The radiation pressure in moving media with axial symmetry was studied [164].

Reviews [165, 166].

7.8 H. Domke

With V.V. Ivanov [104].

The radiative transfer theory in spectral lines was expanded to the presence of a weak magnetic field in [167].

The problem of conservative Rayleigh scattering of polarized radiation in a semi-infinite medium was reduced to searching for several source functions depending only on the optical depth and determined by the integral equations. For the Milne and the reflection problems the number of these functions is equal to two [168]. The results were transferred to a finite layer.

The general scattering matrix was expanded to the generalized spherical functions. The corresponding radiation fields were separated into the azimuthal harmonics [169].

The singular solutions to the equation of the polarized radiation transfer were determinated and the solutions to the multiple scattering problems were expanded in elementary modes [170, 171].

The methods to calculate the transfer of polarized radiation were proposed: the doubling method [172], the application of the invariance principles [173] and of the transfer equation in the Q-form [138] expanded to polarized radiation [174] (coauthor E.G. Yanovitskij).

The transformation of the equation for H-function in order to accelerate the convergence of iterations [175], which was followed by the albedo shifting method.

Book [176].

7.9 S.I. Grachev

With V.V. Ivanov [109] and with V.P. Grinin [162].

The characteristic lengths of radiation transfer in a one-dimensional infinite medium expanding with a constant velocity gradient (the thermalization length, the thickness of the boundary layer, the diffusion length) were determinated. The asymptotic behavior of the solutions were obtained by the factoring method for the rectangular, the Doppler and the power absorption profiles [177].

The asymptotics for the resolvent functions and the source functions for particular source distributions (uniform, exponential, point source) with scattering in an infinite medium isotropically expanding with a constant (small) velocity gradient were deduced. The scattering is considered to be conservative with the Doppler or power absorption profiles [178].

The asymptotic self-similar representations of the kernel and resolvent functions, which characterize the radiation fields in a three-dimensional infinite and semi-infinite media expanding with the velocity gradient were obtained [179].

The explicit expression of the resolvent function was deduced for the problem of the non-stationary line radiation field in a semi-infinite medium for scattering with CFR and $t_2 = 0$ [180] in terms of the eigenfunctions of the basic integral equation (1) found in [146].

The polarization in a spectral line was investigated. In [181] it was shown that the asymptotic expansion for the matrix of the source functions in the problem of the line scattering with CFR and the Doppler profile could be obtained directly from the matrix equation defining it. Some particular cases of the true absorption and the depolarization values were considered. In [182] the problem of calculating the line radiation fields in the medium with uniform distribution of sources was reduced to two nonlinear equations for the matrices of the dimension 6×6. For scattering with CFR (even with the Hanle effect) the two equations were replaced by one. In [183] the asymptotic and numerical solutions to this equation were obtained. Finally, in [184] the Hanle matrix was factorized and the matrix generalization of the so-called $\sqrt{\varepsilon}$ law ($\varepsilon = 1 - \lambda$) was deduced.

Review [185].

8 Conclusion

Thus, V.V. Sobolev and his disciples have succeeded in building the analytical radiative transfer theory for monochromatic scattering as well as for scattering in spectral lines, including the scattering of polarized and non-stationary radiation. Their calculations demonstrated characteristic features of various types of scattering and are in qualitative agreement with the observational data for a variety of astrophysical objects.

Certainly several other groups have been studying the same problems. Their works are described in the reviews mentioned in the text. These groups exchanged the information and results as well as cited the works by each other. In this review we summarize only the main works by V.V. Sobolev and his school.

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Inhomogeneous Semi-Infinite Atmospheres – On Transforming Conservative Multiple Scattering to Non-Conservative Multiple Pseudo-Scattering

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The F- and K-integrals are used to transform the zeroth azimuthal Fourier component of the radiative transfer equation for conservative multiple scattering of polarized light in vertically inhomogeneous plane atmospheres into an equivalent transfer equation with a modified phase matrix corresponding to non-conservative pseudo-scattering. As an example, the transformation to non-conservative multiple pseudo-scattering is applied to express the surface Green's function matrix for conservative scattering in terms of the surface Green's function matrix for non-conservative pseudo-scattering.

1 Introduction

The exclusive property of the transfer equation for conservative multiple scattering, which permits to determine the first and second angular moments of the intensity of the radiation field, the so called F- and K-integrals, a priori, up to two constant parameters, has been pointed out by Chandrasekhar [1] as well as by Sobolev [2] and partly employed by them on treating radiative transfer problems in vertically homogeneous conservative plane media. Here, it is shown, that even for vertically inhomogeneous conservative media, the F- and K-integrals allow us to transform the conservative radiative transfer equation into an equivalent transfer equation of the same form corresponding to non-conservative pseudo-scattering.

2 The transfer equation

Let us consider the transfer of polarized radiation in a vertically inhomogeneous and source-free plane atmosphere with local conservative scattering properties assumed to be macroscopically isotropic and mirror symmetric. It is well known (c.f. [3]) that, after azimuthal Fourier decomposition, the only conservative

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transfer equation emerges for the two-component vector of the azimuthally averaged Stokes parameters I and Q

$$u\frac{\partial}{\partial\tau}\mathbf{I}(\tau,u) = -\mathbf{I}(\tau,u) + \frac{1}{2}\int_{-1}^{+1} dv\mathbf{W}_{IQ}(\tau;u,v)\,\mathbf{I}(\tau,v), \tag{1}$$

where $\tilde{\mathbf{I}}(\tau,u) = (I(\tau,u), \quad Q(\tau,u))$. Here, the tilde denotes transposition of the vector, τ is the optical depth in the atmosphere, and u is the cosine of the polar angle with respect to the inner normal at the top $\tau = 0$ of the atmosphere. The matrix $\mathbf{W}_{IQ}(\tau;u,v)$ is the azimuthally averaged I,Q-component of the complete phase matrix. Local macroscopic mirror symmetry and reciprocity imply [3]

$$\mathbf{W}_{IQ}(\tau;u,v) = \mathbf{W}_{IQ}(\tau;-u,-v) = \tilde{\mathbf{W}}_{IQ}(\tau;v,u), \tag{2}$$

respectively. For conservative scattering, there hold the integral relations

$$\frac{1}{2}\int_{-1}^{+1} dv\,\mathbf{W}_{IQ}(\tau;u,v)\,\mathbf{i}_0 = \mathbf{i}_0, \quad \frac{1}{2}\int_{-1}^{+1} dv\,\mathbf{W}_{IQ}(\tau;u,v)v\,\mathbf{i}_0 = \frac{u}{3}\beta_1(\tau)\,\mathbf{i}_0, \tag{3}$$

where $\tilde{\mathbf{i}}_0 = (1, \quad 0)$. By means of Eq. (1) in conjunction with Eqs. (2), and (3), we find that the flux of radiative energy will be constant, i.e.,

$$F(\tau) = \frac{1}{2}\int_{-1}^{+1} du\,u\,\tilde{\mathbf{i}}_0\,\mathbf{I}(\tau,u) = F = const, \tag{4}$$

and the K-integral is found to be

$$K(\tau) = \frac{1}{2}\int_{-1}^{+1} du\,u^2\,\tilde{\mathbf{i}}_0\mathbf{I}(\tau,u) = K(0) - \left(1 - \frac{\bar{\beta}_1(\tau)}{3}\right)\tau F. \tag{5}$$

Here, $\bar{\beta}_1(\tau)$ is defined as $\bar{\beta}_1(\tau) = \frac{1}{\tau}\int_0^\tau dt\bar{\beta}_1(t)$. Finally, two eigensolutions to the transfer equation (1) can be found

$$\mathbf{i}_0(\tau,u) = \mathbf{i}_0, \quad \mathbf{i}_1(\tau,u) = \left[\left(1 - \frac{\bar{\beta}_1(\tau)}{3}\right)\tau - u\right]\mathbf{i}_0. \tag{6}$$

3 The equivalent transfer equation

On defining a modified phase matrix

$$\mathbf{W}_c(\tau;u,v) = \mathbf{W}_{IQ}(\tau;u,v) - \left[c_1(\tau)\,u^2\,\mathbf{i}_0\,\tilde{\mathbf{i}}_0\,v^2 + c_2(\tau)\,u\,\mathbf{i}_0\,\tilde{\mathbf{i}}_0\,v\right], \tag{7}$$

and replacing the phase matrix in Eq. (1) by means of Eq. (7), and using also Eqs. (4) and (5), we rewrite the conservative transfer equation (1) in the form

$$\begin{aligned} u\frac{\partial}{\partial\tau}\mathbf{I}(\tau,u) &= -\mathbf{I}(\tau,u) + \frac{1}{2}\int_{-1}^{+1} dv\,\mathbf{W}_c(\tau;u,v)\,\mathbf{I}(\tau,v) \\ &+ c_1(\tau)\,u^2\,\mathbf{i}_0\left[K(0) - \left(1 - \frac{\bar{\beta}_1(\tau)}{3}\right)\tau F\right] + c_2(\tau)\,u\,\mathbf{i}_0 F. \end{aligned} \tag{8}$$

Obviously, the new transfer equation (8) describes non-conservative multiple pseudo-scattering, with some primary (pseudo-) source terms on the r.h.s. linearly dependent on two constants F and $K(0)$, which can be determined a posteriori. We note that a particular solution to the transfer equation (8) can be found in terms of the eigensolutions (6) of the original conservative transfer equation (1)

$$\mathbf{I}_p(\tau, u) = 3\left[\mathbf{i}_0 K(0) - \mathbf{i}_1(\tau, u) F\right]. \tag{9}$$

4 Semi-infinite medium surface Green's function matrix

The semi-infinite medium surface Green's function matrix $\mathbf{G}(\tau, u; 0, \mu_0)$, with $-1 \leq u \leq +1$, $0 \leq \mu_0 \leq 1$, and $0 < \tau < \infty$, is defined as the finite solution to the transfer equation

$$u\frac{\partial}{\partial \tau}\mathbf{G}(\tau, u; 0, \mu_0) = -\mathbf{G}(\tau, u; 0, \mu_0) + \frac{1}{2}\int_{-1}^{+1} dv\, \mathbf{W}(\tau; u, v)\, \mathbf{G}(\tau, v; 0, \mu_0), \tag{10}$$

subject to the half-range boundary condition

$$\mathbf{G}(+0, \mu; 0, \mu_0) = \frac{1}{\mu}\delta(\mu - \mu_0)\,\mathbf{E}, \quad \mu, \mu_0 \in [0, 1], \tag{11}$$

at the top, where $\mathbf{E} = \mathbf{diag}(1, 1)$. In terms of the surface Green's function, the matrix of diffuse reflection is given by

$$\mathbf{R}(\mu, \mu_0) = \frac{1}{2}\mathbf{G}(+0, -\mu; 0, \mu_0), \quad \mu, \mu_0 \in [0, 1], \tag{12}$$

where μ_o denotes the direction of incidence. Reciprocity implies $\mathbf{R}(\mu, \mu_0) = \tilde{\mathbf{R}}(\mu_0, \mu)$ (c.f. [3]). There is no net flux of radiative energy for finite radiation fields in a semi-infinite conservatively scattering atmosphere without internal primary sources. Thus, the F-integral of the corresponding surface Green's function matrix $\mathbf{G}_{IQ}(\tau, u; 0, \mu_0)$ becomes zero,

$$\tilde{\mathbf{F}}_{IQ}(\tau; 0, \mu_0) = \tilde{\mathbf{F}}_{IQ}(\tau; 0, \mu_0) = \frac{1}{2}\int_{-1}^{+1} du\, u\, \tilde{\mathbf{i}}_0\, \mathbf{G}_{IQ}(\tau, u; 0, \mu_0) = \mathbf{0}. \tag{13}$$

Instead of seeking the surface Green's function matrix $\mathbf{G}_{IQ}(\tau, u; 0, \mu_0)$ as the solution to the conservative transfer equation (10) with $\mathbf{W}(\tau; u, v) = \mathbf{W}_{IQ}(\tau; u, v)$, we apply the equivalent transfer equation (8) corresponding to non-conservative pseudo-scattering, where $\mathbf{I}(\tau, u)$ is replaced by the function matrix $\mathbf{G}_{IQ}(\tau, u; 0, \mu_0)$, while $F = 0$, and $K(0)$ is replaced by the transposed vector

$$\tilde{\mathbf{K}}_{IQ}(+0; 0, \mu_0) = \frac{1}{2}\int_{-1}^{+1} du\, u^2\, \tilde{\mathbf{i}}_0\, \mathbf{G}_{IQ}(+0, u; 0, \mu_0). \tag{14}$$

On taking into account the particular solution (9), we use the surface Green's function matrix $\mathbf{G}_c(\tau, u; 0, \mu_0)$ for non-conservative pseudo-scattering to get, after some algebra, the surface Green's function matrix for conservative scattering as

$$\mathbf{G}_{IQ}(\tau, u; 0, \mu_0) = \mathbf{G}_c(\tau, u; 0, \mu_0) + \frac{3}{D}\left[\mathbf{i}_0 - \int_0^1 d\eta\, \mathbf{G}_c(\tau, u; 0, \eta)\eta\, \mathbf{i}_0\right]\tilde{\mathbf{K}}_c(+0; 0, \mu_0) \tag{15}$$

with

$$\tilde{\mathbf{K}}_c(+0; 0, \mu_0) = \frac{1}{2}\left[\mu_0\, \tilde{\mathbf{i}}_0 + 2\int_0^1 d\mu\, \mu^2\, \tilde{\mathbf{i}}_0\, \mathbf{R}_c(\mu, \mu_0)\right], \tag{16}$$

and $D = 3\int_0^1 d\eta\, \tilde{\mathbf{K}}_c(+0; 0, \eta)\, \eta\, \mathbf{i}_0$, while $\tilde{\mathbf{K}}_c(+0; 0, \mu_0) = D\, \tilde{\mathbf{K}}_{IQ}(+0; 0, \mu_0)$. It is easy to verify that $\mathbf{G}_{IQ}(\tau, u; 0, \mu_0)$ as given by Eq. (15) satisfies the correct transfer equation (8) as well as the boundary condition (11). When specified with $\tau = +0$ and $u = -\mu$, Eq. (15) provides a simple formula for retrieving the reflection matrix $\mathbf{R}_{IQ}(\mu, \mu_0)$ for conservative scattering by means of the reflection matrix $\mathbf{R}_c(\mu, \mu_0)$ for non-conservative pseudo-scattering

$$\mathbf{R}_{IQ}(\mu, \mu_0) = \mathbf{R}_c(\mu, \mu_0) + \frac{3\,(1 - D)}{D^2\gamma}\, \mathbf{K}_c(+0; 0, \mu)\, \tilde{\mathbf{K}}_c(+0; 0, \mu_0), \tag{17}$$

where the constant $\gamma = 3\int_0^1 d\eta\, \eta^2\, \tilde{\mathbf{i}}_0\, \mathbf{K}_{IQ}(+0; 0, \eta) = \frac{3}{D}\int_0^1 d\eta\, \eta^2\, \tilde{\mathbf{i}}_0\, \mathbf{K}_c(+0; 0, \eta)$ is the so called extrapolation length well known in radiative transfer theory.

For practical methods to calculate reflection matrices for inhomogeneous semi-infinite atmospheres, which are applicable also to compute $\mathbf{R}_c(\mu, \mu_0)$ for non-conservative pseudo-scattering, we refer to the textbook of Yanovitsky [4] and references therein. Finally, we note that for homogeneous atmospheres the transformation to equivalent pseudo-scattering with reduced effective albedo of single scattering can be performed also for non-conservative scattering. This has been described in an earlier paper [5].

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Bilinear Expansions for Redistribution Functions

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We suggest here a method for construction of a bilinear expansion for an angle-averaged redistribution function. An eigenvalues and eigenvectors determination problem is formulated and the required matrices are found analytically, and numerical procedures for their computations are elaborated. A simple method for the accuracy evaluation of the numerical calculations is suggested. It is shown that a group of redistribution functions describing the light scattering process within the spectral line frequencies can be constructed if the eigenvalue problem is solved for the considered function. It becomes possible if various combinations of eigenvalues and eigenvectors with the basic functions are used.

1 The redistribution function $r_{II}(x', x)$

Let us first redefine the redistribution function $r(x', x)$ which has a rather simple physical meaning: the quantity $r(x', x)dx$ represents the probability that a photon with the dimensionless frequency x' will be absorbed by an atom and re-emitted then in the frequency interval $(x; x + dx)$. The introduced dimensionless frequencies show the distance of photon's frequency $\nu(\nu')$ from the line center frequency ν_0 in Doppler half widths $\left(x = \frac{\nu-\nu_0}{\Delta\nu_D}\right)$. This redistribution function differs from one defined by Hummer [1] by the constant factor $\left(\pi^{\frac{1}{4}} U(0, \sigma)\right)^{-1}$, where the function

$$U(x, \sigma) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-t^2)}{(x - t)^2 + \sigma^2} dt \tag{1}$$

is the well known Voigt function and $\sigma = \frac{\Delta\nu_T}{\Delta\nu_D}$, where $\Delta\nu_T$ is the total half-width of the line caused by all the broadening mechanisms taken into account.

The redistribution function describing the photon scattering within the line frequencies of the model two-level atom the upper level of which is broadened due to radiation damping has been independently derived by Henyey [2], Unno [3] and Sobolev [4] assuming that in the atom's reference frame the scattering is coherent.

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Then, using also Hummer's [1] designation, one can represent it in the following form:

$$r_{II}(x', x) = \frac{1}{\pi U(0,\sigma)} \int_{\frac{|\overline{x}-\underline{x}|}{2}}^{\infty} \exp(-t^2) \left[\arctan \frac{\underline{x}+t}{\sigma} - \arctan \frac{\overline{x}-t}{\sigma} \right] dt. \qquad (2)$$

In the expression (2) we used the following denotations: $\overline{x} = \sup(x', x)$ and $\underline{x} = \inf(x', x)$.

It is noteworthy that there has been known bilinear expansion for two out of four redistribution functions described in Hummer's paper [1], namely, $r_I(x', x)$ and $r_{III}(x', x)$ before their classification by him. This fact was rather important for solving the light scattering problems applying the Principle of Invariance (PI). However, up to nowadays no any "natural" bilinear expansion has been revealed for the function $r_{II}(x', x)$. Therefore, one might try to create such a bilinear expansion using some artificial procedures.

In order to construct numerically such an expansion, let us first introduce here another representation of $r_{II}(x', x)$ derived by Nikoghossian [5] (see also Heinzel's paper [6])

$$r_{II}(x', x) = \frac{\sigma}{\pi U(0,\sigma)} \int_{-\infty}^{\infty} \frac{r_I(x'+t, x+t)}{t^2+\sigma^2} dt. \qquad (3)$$

From Eq. (3) one finds easily that the function $r_{II}(x', x)$ transforms into the $r_I(x', x)$ when $\sigma = 0$.

On the other hand, the function $r_I(x', x)$ allows the following bilinear expansion first derived by Unno [7]:

$$r_I(x', x) = \int_{|\overline{x}|}^{\infty} \exp(-t^2) dt = \sum_{k=0}^{\infty} \frac{\alpha_{2k}(x')\, \alpha_{2k}(x)}{2k+1}, \qquad (4)$$

where

$$\alpha_k(x) = (2^k \pi^{\frac{1}{2}} k!)^{-\frac{1}{2}}\, H_k(x) \exp(-x^2) \qquad (5)$$

and $H_k(x)$ are the Hermit polynomials.

The obvious connection between functions $r_{II}(x', x)$ and $r_I(x', x)$ expressed by relation (3) allows suggesting the functions (5) as basic ones for constructing the eigenfunctions of $r_{II}(x', x)$. Taking into account this connection, one can search for the bilinear expansion of $r_{II}(x', x)$ in the following form:

$$r_{II}(x', x) = \sum_{k=0}^{\infty} \frac{\omega_{2k}(x', \sigma)\, \omega_{2k}(x, \sigma)}{\zeta_k(\sigma)}, \qquad (6)$$

where

$$\omega_{2k}(x, \sigma) = \sum_{m=0}^{\infty} \gamma_{km}(\sigma)\, \alpha_{2k}(x). \qquad (7)$$

The vector $\zeta_k(\sigma)$ and matrix $[\gamma_{km}(\sigma)]$ are, respectively, the eigenvalues and eigenfunctions of the following problem (see, for example, [8, 9]):

$$\sum_{m=0}^{\infty} [\gamma_{km}(a_{mn} - \zeta_k(\sigma) b_{mn})] = 0, \tag{8}$$

where

$$a_{mn} = \int_{-\infty}^{\infty} \alpha_{2m}(x)\, \alpha_{2n}(x) dx, \tag{9}$$

and

$$b_{mn} = \int_{-\infty}^{\infty} \alpha_{2m}(x) dx \int_{-\infty}^{\infty} r_{II}(x', x)\, \alpha_{2n}(x') dx'. \tag{10}$$

It is evident that calculating the matrices $[a_{mn}]$ and $[b_{mn}]$ and solving the eigenvalue problem (8) one can numerically construct the bilinear expansion (6).

2 Calculation of the relevant matrices

Using the integral forms for the Hermit polynomials, one can easily find the following presentation for the introduced above basic functions [10]:

$$\alpha_k(x) = (2^k \pi^{\frac{1}{2}} k!)^{-\frac{1}{2}} \frac{2}{\sqrt{\pi}} \operatorname{Re}(-2i)^k \int_0^{\infty} t^k \exp(-t^2 + 2ixt) dt. \tag{11}$$

Then, using the following expression for the δ function:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ixt) dt = \delta(t), \tag{12}$$

one finds directly

$$a_{mn} = (-1)^{m+n} \frac{(2m + 2n - 1)!!}{2^{m+n+\frac{1}{2}} \sqrt{(2m)!(2n)!}}. \tag{13}$$

For calculations of the matrix $[b_{mn}]$, one can suggest two different ways. One is the direct calculation of the threefold iterated integral (10) which is fraught with huge numerical difficulties arising due to the complicated behavior of the basic functions. Therefore, some simplifying analytical calculations before starting the numerical procedures would sufficiently facilitate the numerical procedures. One can find from Eq. (10) substituting Eq. (4) for the redistribution function $r_I(x', x)$ in the relation (3)

$$b_{mn} = \frac{\sigma}{\pi U(0, \sigma)} \sum_{k=0}^{\infty} \frac{1}{2k+1} \int_{-\infty}^{\infty} \frac{g_{km}(t)\, g_{kn}(t)}{t^2 + \sigma^2} dt, \tag{14}$$

where

$$g_{km}(t) = \int_{-\infty}^{\infty} \alpha_{2k}(x + t)\, \alpha_{2m}(x) dx = N_{km}\, \alpha_{km}\left(\frac{t}{\sqrt{2}}\right), \tag{15}$$

and

$$N_{km} = \frac{\pi^{\frac{1}{4}}}{2^{k+m+\frac{1}{2}}} \sqrt{\frac{(2k+2m)!}{(2k)!(2m)!}}. \tag{16}$$

Thus, one finds finally

$$b_{mn} = \frac{1}{U(0,\sigma)} \sum_{k=0}^{\infty} \frac{N_{km} N_{kn}}{2k+1} c_{k+m,k+n}, \tag{17}$$

where

$$c_{mn} = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\alpha_{2m}\left(\frac{t}{\sqrt{2}}\right) \alpha_{2n}\left(\frac{t}{\sqrt{2}}\right)}{t^2+\sigma^2} dt. \tag{18}$$

As a matter of fact, the threefold iterated integral is given now by an infinite series where only a single integration appears. However, the integrand is again a vastly oscillating function making the direct numerical computation extremely inefficient especially for greater values of indexes. Also it is not difficult to realize that for the smaller damping parameters the computing error gets larger. But at the same time in the limiting case when $\sigma = 0$, the integral (18) can be taken analytically to find

$$c_{mn}|_{\sigma=0} = \alpha_{2m}(0)\, \alpha_{2n}(0). \tag{19}$$

In order to calculate the integral (18) for the values $\sigma > 0$, let us use the formulae (5) and the Hermit polynomials definition (see, for example, [10])

$$H_{2k}(x) = (2k)! \sum_{l=0}^{k} \frac{(-1)^l}{(l)!(2k-2l)!} (2x)^{2k-2l}, \tag{20}$$

to obtain

$$\alpha_{2n}\left(\frac{t}{\sqrt{2}}\right) = \exp\left(-\frac{t^2}{2}\right) \frac{\sqrt{(2n)!}}{\pi^{\frac{1}{4}}} \sum_{k=0}^{n} \frac{(-1)^k t^{2n-2k}}{2^k k!(2n-2k)!}. \tag{21}$$

Then, taking into account that

$$t^{2k} = \frac{(2m)!}{2^{2m}} \sum_{j=0}^{m} \frac{H_{2j}(t)}{(2j)!\,(m-j)!}, \tag{22}$$

one can finally find

$$c_{mn} = \frac{\sqrt{(2m)!(2n)!}}{\pi^{\frac{1}{4}} 2^{2m+2n}} \sum_{k=0}^{m} \frac{(-2)^k}{k!\,(2m-2k)!} \sum_{l=0}^{n} \frac{(-2)^l}{l!\,(2n-2l)!}$$

$$\times\, (2m+2n-2k-2l)! \sum_{q=0}^{m+n-k-l} \frac{2^q \alpha_{2q}(0,\sigma)}{(m+n-k-l-q)!\sqrt{(2q)!}}, \tag{23}$$

where the following notation is introduced:

$$\alpha_{2q}(0,\sigma) = \frac{(-1)^q}{\pi^{\frac{1}{4}}\sqrt{(2q)!}} \sum_{p=0}^{\infty} \frac{2^p \sigma^{2p}}{(2p)!} \left[(2q+2p-1)!! - \frac{\sigma}{\sqrt{\pi}} \frac{2}{2p+1} (2q+2p)!! \right]. \quad (24)$$

The expression (23) obtained for description of elements of the required matrix, though explicit, is again rather complicated for direct numerical calculations. Therefore, any numerical procedure based on the ordinary accuracy of the used computer calculations cannot provide the required accuracy of the final results. These difficulties can be overcome only using methods of calculations based on the usage of a higher number of significant digits. For example, about one hundred twenty or more significant digits are needed to provide 15 correct digits for all the elements of the 100×100 matrix.

Nevertheless, it is possible to obtain a much simpler expression if one of the indexes of the matrix $[c_{mn}]$ is equal to zero (the first row or the first column). Then one out of the three sums disappears immediately and one obtains after some transformations

$$c_{0,n} = \frac{(-1)^n \sqrt{(2n)!}}{\pi^{\frac{1}{4}} 2^{2n}} \sum_{q=0}^{n} \frac{(-2)^q \, \alpha_{2q}(0,\sigma)}{(n-q)! \sqrt{(2q)!}} = c_{n,0}. \quad (25)$$

On the other hand, taking into account the relation of recurrence for the Hermit polynomials

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad (26)$$

one can derive the following recurrence relation for the required elements of the matrix $[c_{mn}]$:

$$c_{mn} = \sqrt{\frac{2n+1}{2m}} d_{m-1,n+1} + \sqrt{\frac{n}{m}} d_{m-1,n-1} - \sqrt{\frac{2m-1}{2m}} c_{m-1,n}, \quad (27)$$

where

$$d_{mn} = c_{m+\frac{1}{2},n+\frac{1}{2}}. \quad (28)$$

Further, in terms of the physical meaning of the redistribution function one might conclude that its integral over one of the arguments should give the profile of the absorption coefficient

$$\int_{-\infty}^{\infty} r_{II}(x',x)dx' = \alpha(x,\sigma) = \frac{U(x,\sigma)}{U(0,\sigma)}, \quad (29)$$

and bearing in mind (5)–(7), one finds

$$\sum_{k=0}^{\infty} \frac{\gamma_{k,0}}{\zeta_k} \, \omega_{2k}(x,\sigma) = \frac{U(x,\sigma)}{U(0,\sigma)}. \quad (30)$$

Here the following normalization relation is used:

$$\int_{-\infty}^{\infty} \alpha_m(x)\,\alpha_n(x)dx = \delta_{mn}, \tag{31}$$

where δ_{mn} is the Kronecker symbol. Integrating Eq. (34) over all frequencies, one obtains finally

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} r_{II}(x', x)dx' = \sum_{k=0}^{\infty} \frac{{\gamma_{k0}}^2}{\zeta_k} = \sqrt{\pi}, \tag{32}$$

which can be used for the normalization purposes.

Now let us briefly consider the physical situation when both energetic levels are broadened. Heinzel [6] has shown that the redistribution function derived by Hummer [1] for description of this process is not correct and obtained a new expression allowing the following notation:

$$r_V(x', x) = \frac{{\sigma_i}^2}{\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t^2 + {\sigma_i}^2} \int_{-\infty}^{\infty} \frac{r_{II}(x' + t, x + u)}{u^2 + {\sigma_i}^2} du. \tag{33}$$

Then, using Eq. (6), one will find a bilinear expansion for this function as well. Putting Eq. (6) into Eq. (33), one obtains

$$r_V(x', x) = \sum_{k=0}^{\infty} \frac{\omega_{2k}(x', \sigma_i, \sigma_j)\,\omega_{2k}(x, \sigma_i, \sigma_j)}{\zeta_k(\sigma_j)}, \tag{34}$$

where the functions

$$\omega_{2k}(x, \sigma_i, \sigma_j) = \sum_{m=0}^{\infty} \gamma_{km}(\sigma_j)\,\alpha_{2m}(x, \sigma_i) \tag{35}$$

depend on damping parameters of both energetic levels. The functions $\alpha_{2k}(x, \sigma)$ are defined by the relation

$$\alpha_k(x, \sigma) = (2^k \pi^{\frac{1}{2}} k!)^{-\frac{1}{2}} \frac{2}{\sqrt{\pi}} \mathrm{Re}(-2i)^k \int_0^{\infty} t^k \exp(-t^2 - 2\sigma t + 2ixt)dt. \tag{36}$$

Thus, constructing a bilinear expansion for the function $r_{II}(x', x)$ as described above, one arrives at a conclusion that this method provides a tool for constructing similar expansions for all the applicable redistribution functions. It can be done immediately, if one obtains the eigenfunctions $\gamma_{km}(\sigma)$ and eigenvalues $\zeta_k(\sigma)$ and also uses an appropriate numerical procedure for computing the functions $\alpha_k(x, \sigma)$. Then the corresponding redistribution functions could be constructed by the same procedure using the various values of the parameters σ_i and σ_j. It is easy to see that $r_V(x', x) = r_{III}(x', x)$, if $\sigma_j = 0$, $r_V(x', x) = r_{II}(x', x)$ for $\sigma_i = 0$ and, at last, $r_V(x', x) = r_I(x', x)$, if both damping parameters are equal to zero – $\sigma_j = \sigma_i = 0$.

3 The auxiliary functions $\alpha_k(x, \sigma)$

Obviously, besides the eigenvalue problem (8) one should overcome the second key computational difficulties for the eventual construction of the redistribution functions. That is the problem of the numerical evaluation of the corresponding auxiliary functions. The functions $\alpha_{2m}(x, \sigma)$ defined by Eq. (36) have been introduced and studied by Hummer [1], and a rather effective method for their calculation was suggested by him in the same paper. In order to simplify the initial expression (36), the exponent $\exp(-2\sigma t)$ is replaced by its power series. Then one should compute several terms of that series to provide the required accuracy of auxiliary functions. Following the Hummer's procedure in general, Harutyunian [11] has separated from each other the even and odd functions appearing in the derived series to obtain the following relation:

$$\alpha_k(x, \sigma) = (2^k \pi^{\frac{1}{2}} k!)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(i\sigma)^m}{(2m)!} \left[M_{k+2m}(x) + \frac{\sigma}{2m+1} N_{k+2m+1}(x) \right], \quad (37)$$

where

$$M_k(x) = \frac{2}{\sqrt{\pi}} \operatorname{Re}(-2i)^k \int_0^{\infty} t^k \exp(-t^2 + 2ixt) dt \quad (38)$$

and

$$N_k(x) = \frac{2}{\sqrt{\pi}} \operatorname{Im}(-2i)^k \int_0^{\infty} t^k \exp(-t^2 + 2ixt) dt \quad (39)$$

are the Hermit functions of the first and second kinds [10].

From Eqs. (38) and (39) one can easily find the following recurrent formulas well known from the mathematical textbooks (see, for example, [10]):

$$M_{k+1}(x) = 2x M_k(x) - 2k M_{k-1}(x) \quad (40)$$

for the first kind functions and similarly

$$N_{k+1}(x) = 2x N_k(x) - 2k N_{k-1}(x) \quad (41)$$

for the second kind functions. The first functions to be used for recurrent relations are defined as follows:

$$M_0(x) = \exp(-x^2), \quad M_1(x) = 2x M_0(x), \quad (42)$$

$$N_0(x) = \frac{2}{\sqrt{\pi}}, \quad N_1(x) = 2x N_0(x) - \frac{2}{\sqrt{\pi}}. \quad (43)$$

Here

$$F(x) = \int_0^{\infty} \exp(-t^2) \sin 2xt \, dt = \exp(-x^2) \int_0^{x} \exp(t^2) dt \quad (44)$$

is the Dawson function connected with the error function of an imaginary argument and represents the solution of the following Cauchy problem:

$$F'(x) = 1 - 2xF(x) \tag{45}$$

with the initial condition $F(0) = 0$.

Numerical procedures for calculation of the Dawson function are considered in Hummer's paper [12]. Some earlier references could be found in the mentioned above review by Hummer [1]. Among the relatively recent studies one might refer to the papers [13–14]. The most efficient procedure for calculation of the Dawson function can be carried out using the power series [10]

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, 2^n}{(2n+1)!!} x^{2n+1}, \tag{46}$$

which converges for all values of the argument. However, one should take care for the accuracy issues when applying the relation (46) for numerical computations. Obviously, for the smaller values of the argument ($x \leq 1$) the series (46) converges rather rapidly and no big difficulties can arise. However, for the larger values of the argument, the need in much higher digit numbers for calculations grows up very rapidly. For instance, for $x = 12$ one can easily provide around 35 correct digits of the Dawson function if uses 120 significant digits for calculations. Nonetheless, the usage of the same number of significant digits provides only 12 correct digits in the final result if the argument reaches to the value $x = 15$. Many correct significant digits are very important not only for computing the Dawson function itself. The point is that the recurrent formula themselves are a perilous source of the error accumulation and therefore one needs to calculate the Dawson function with a bigger number of correct significant digits. Actually, the problem is absolutely the same that we encountered considering the matrix $[c_{mn}]$ in the previous paragraph.

Of course, on the other hand, one can find an asymptotic series for the larger arguments of the Dawson function which can be rather useful for the practical applications [10]

$$F(x) \approx \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n+1} \, n! \, x^{2n+1}}. \tag{47}$$

This asymptotic relation, as opposed to the series (46), is a diverging one. Nevertheless, a few first terms of this series will provide an applicable accuracy for various asymptotic estimates. Indeed, starting with the relations (42)–(43) and using the relation (47), one obtains for $x \to \infty$ the following asymptotic form:

$$N_k(x) \approx \frac{(-1)^k}{\sqrt{\pi} x^{k+1}} \sum_{n=0}^{\infty} \frac{(2n+k)!}{2^{2n} \, n! \, x^{2n}}, \tag{48}$$

which can be used in the series (37). It is easy to see that due to the exponentially decreasing behavior of the first kind Hermit functions for larger values of the argument they are falling much faster than the second kind functions. Therefore, one finds the asymptotic relation

$$\alpha_k(x,\sigma) = \frac{\sigma}{x^{k+2}\sqrt{\pi}}(2^k\pi^{\frac{1}{2}}k!)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2n+k+1)!}{x^{2n}} \sum_{m=0}^{n} \frac{(-1)^m\sigma^{2m}}{2^{2(n-m)}(2m+1)!(n-m)!}, \tag{49}$$

which turns into the known asymptotic expression for the Voigt function [15]

$$U(x,\sigma) = \frac{\sigma}{x^2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(2n+1)!}{x^{2n}} \sum_{m=0}^{n} \frac{(-1)^m\sigma^{2m}}{2^{2(n-m)}(2m+1)!(n-m)!}. \tag{50}$$

These asymptotic forms coupled with the exact formulas derived above provide one with all the necessary tools for building the bilinear expansions of redistribution functions and their usage for the practical purposes.

Preliminary calculations show that these numerical procedures easily can be performed on modern PC. Elaborated specially for these purposes software package HAHMATH allows one to perform computations with the needed number of significant digits when high accuracy calculations are required. However, extraordinary accuracies are needed only when the matrix $[c_{mn}]$ or Dawson function and its derivatives are calculated. Once calculated the matrix $[c_{mn}]$ can be used for building the matrix $[b_{mn}]$ and to continue all other computations with the ordinary accuracy of computers. There is no need for using the extremely long numbers when solving the corresponding eigenvalue problem. Calculated once the eigenvalues and eigenfunctions for the given damping factor might be used for further calculations.

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About the Development of the Asymptotic Theory of Non-Stationary Radiative Transfer

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A brief review of the development of the asymptotic non-stationary radiative transfer theory is presented. In particular, the accuracy of the diffusion approximation is studied. It is shown that the replacement of the non-stationary transfer equation by the heat conductive equation should give satisfactory results when the single scattering albedo λ is close to the unity. But this approximation can lead to significant errors when $\lambda < 1$.

Studying time-dependent processes in various non-stationary objects is an important problem of modern astrophysics. The illumination of the dust nebula under the influence of radiation of a new star can be considered as an example of such process.

Sobolev initiated the systematic development of the theory of non-stationary radiation fields in the article [1] published in 1952. Fundamentals of this theory were presented in his book [2].

Non-stationary radiation fields are characterized by the finite speed of light c and a definite duration of the light scattering process.

Let t_1 be the mean time of the stay of a photon in the absorbed state. It is usually assumed that the probability of emission of a photon being in the absorbed state in the time interval from t to $t + dt$ depends on t by the exponential law, i.e., it is proportional to $e^{-\frac{t}{t_1}} \frac{dt}{t_1}$.

The probability of the photon absorption while travelling after his radiation during an interval of time from t to $t + dt$ depends on t also exponentially, e.g., it is proportional to $e^{-\frac{t}{t_2}} \frac{dt}{t_2}$, where $t_2 = \frac{1}{\alpha c}$ is the mean time of stay of a photon on the path between two consecutive scatterings. Here α is the volume absorption coefficient of the medium.

The values of t_1 and t_2 are usually very different from each other. Therefore, Sobolev has proposed to allocate the consideration of two limiting cases, i.e., the case A, when $t_1 \gg t_2$, and the case B, when $t_2 \gg t_1$.

The simplest model of non-stationary radiative transfer is a model based on the consideration of the one-dimensional homogeneous infinite medium with an energy source depending on time. Let us assume that the medium is illuminated by a momentary point source of luminosity L flashing at some initial moment of time. We note that an actual flash duration and a dependence of the luminosity $L(t)$

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on the time t can be taken into account by integrating over time the expressions for light field characteristics found in the case of a point source multiplied by the value of the luminosity $L(t)$.

Let $I_1(r,t)$ and $I_2(r,t)$ be intensities of the radiation spreading on distance r from the source at time t in the direction of increasing and decreasing values of the coordinate r, respectively. Instead of the geometric distances r, the physical time t and values t_1 and t_2, we use the corresponding dimensionless quantities

$$\tau = \alpha r, \quad u = \frac{t}{t_1 + t_2}, \quad \beta_1 = \frac{t_1}{t_1 + t_2}, \quad \beta_2 = \frac{t_2}{t_1 + t_2}. \tag{1}$$

Then the radiative transfer equation takes the following form:

$$\frac{\partial I_1(\tau,u)}{\partial \tau} + \beta_2 \frac{\partial I_1(\tau,u)}{\partial u} = -I_1(\tau,u) + B(\tau,u), \tag{2}$$

$$-\frac{\partial I_2(\tau,u)}{\partial \tau} + \beta_2 \frac{\partial I_2(\tau,u)}{\partial u} = -I_2(\tau,u) + B(\tau,u). \tag{3}$$

Here $B(\tau,u)$ is the source function defined by the equation of radiative equilibrium

$$B(\tau,u) = \frac{\lambda}{2}\int_0^u \left[I_1\left(\tau,u'\right) + I_2(\tau,u')\right] e^{-\frac{u-u'}{\beta_1}} \frac{du'}{\beta_1}, \tag{4}$$

where λ is the single scattering albedo. These equations are supplemented with the initial condition which takes into account the momentary point source of energy. The mean radiation intensity $J(\tau,u)$ and the radiation flux $H(\tau,u)$ are defined by the expressions

$$J(\tau,u) = \frac{1}{2}\left[I_1(\tau,u) + I_2(\tau,u)\right], \tag{5}$$

$$H(\tau,u) = I_1(\tau,u) - I_2(\tau,u). \tag{6}$$

Minin [3] obtained the exact solution of this problem by means of the Laplace transform.

Simple asymptotic expressions for characteristics of the non-stationary radiation field are obtained in the case when points of the medium are located at large optical distances from energy sources ($\tau \gg 1$) and scattering of light is close to conservative ($1-\lambda \ll 1$). In this case Minin [4] proposed to use a simple technique for inverting the Laplace transform. As it is known from the theory of the Laplace transform, the value of the original at large values of the argument ($u \gg 1$) is determined using the expansion of the image in powers of the small parameter s. This expansion corresponds to the expansion of solutions of the stationary radiative transfer equation in powers of the small values of $1-\lambda$. As a result of the Laplace transform in time, the non-stationary equation is converted into the stationary one but the value of λ is replaced by the value $\frac{\lambda}{(1+\beta_1 s)(1+\beta_2 s)}$. Therefore, taking into account the fact that $\beta_1 + \beta_2 = 1$, we obtain $1 - \lambda = s$ with accuracy to members of the higher degrees of the parameter s. Hence, when receiving the asymptotic image, it is necessary to replace the small values of $1 - \lambda$ by s in the equation solution for the stationary case, and then to apply the inverse Laplace transform.

In the case of one-dimensional infinite medium illuminated by a momentary point source, we obtain for $J(\tau, u, \lambda)$ and $H(\tau, u, \lambda)$ the following expressions (for $\lambda = 1$, $\tau \gg 1$, $u > \tau$):

$$J_D(\tau, u, 1) = \frac{L}{4\sqrt{\pi u}}\, e^{-\frac{\tau^2}{4u}}, \tag{7}$$

$$H_D(\tau, u, 1) = \frac{L}{4\sqrt{\pi u}}\, \frac{\tau}{u}\, e^{-\frac{\tau^2}{4u}}. \tag{8}$$

The same expressions for these quantities are obtained in the diffusion approximation in the case of $\lambda = 1$. This approximation is based on using the heat conductivity equation

$$\frac{\partial^2 J(\tau, u, \lambda)}{\partial \tau^2} = \frac{\partial J(\tau, u, \lambda)}{\partial u} + (1 - \lambda)\, J(\tau, u, \lambda) \tag{9}$$

instead of the non-stationary radiation transfer equation. The solution of the equation (9) leads to the following expressions for the functions $J(\tau, u, \lambda)$ and $H(\tau, u, \lambda)$:

$$J_D(\tau, u, \lambda) = e^{-(1-\lambda)u} J_D(\tau, u, 1), \tag{10}$$

$$H_D(\tau, u, \lambda) = e^{-(1-\lambda)u} H_D(\tau, u, 1), \tag{11}$$

where $J_D(\tau, u, 1)$ and $H_D(\tau, u, 1)$ are given by the expressions (7) and (8).

The diffusion approximation was proposed by Compton [5] in 1923. However, in 1926 Milne [6] showed that the usage of this approximation for the calculation of non-stationary fields of radiation can lead to physically unreasonable results.

Kolesov and Sobolev [7] studied the accuracy of the diffusion approximation in the cases A and B.

Exact expressions for $J(\tau, u, \lambda)$ and $H(\tau, u, \lambda)$ in the case A have the form

$$J_A(\tau, u, \lambda) = \frac{L}{2\pi} \int_0^\infty e^{-\left(1 - \frac{\lambda}{1+x^2}\right)u}\, \frac{\cos x\tau}{1 + x^2}\, dx, \tag{12}$$

$$H_A(\tau, u, \lambda) = \frac{L}{\pi} \int_0^\infty e^{-\left(1 - \frac{\lambda}{1+x^2}\right)u}\, \frac{x \sin x\tau}{1 + x^2}\, dx. \tag{13}$$

When $\lambda u \gg 1$, we have the following asymptotic expressions:

$$J_A^{as}(\tau, u, \lambda) \approx \frac{L}{4\sqrt{\pi \lambda u}}\ e^{-(1-\lambda)u - \frac{\tau^2}{4\lambda u}}, \tag{14}$$

$$H_A^{as}(\tau, u, \lambda) \approx \frac{L}{4\sqrt{\pi \lambda u}}\ \frac{\tau}{\lambda u} e^{-(1-\lambda)u - \frac{\tau^2}{4\lambda u}}. \tag{15}$$

In the absence of true absorption, when $\lambda = 1$, these expressions coincide with the expressions (7) and (8) of the diffusion approximation.

In the case B for $\tau \geq 0$ and $u > \tau$, the exact expressions for these quantities are given by the expressions

$$J_B(\tau, u, \lambda) = \frac{\lambda L}{8}\left[I_0\left(\frac{\lambda}{2}\sqrt{u^2-\tau^2}\right) + \frac{u}{\sqrt{u^2-\tau^2}} I_1\left(\frac{\lambda}{2}\sqrt{u^2-\tau^2}\right)\right] e^{-\left(1-\frac{\lambda}{2}\right)u}, \tag{16}$$

$$H_B(\tau, u, \lambda) = \frac{\lambda L}{4}\frac{\tau}{\sqrt{u^2-\tau^2}} I_1\left(\frac{\lambda}{2}\sqrt{u^2-\tau^2}\right) e^{-\left(1-\frac{\lambda}{2}\right)u}, \tag{17}$$

where $I_0(z)$ and $I_1(z)$ are the modified Bessel functions. The asymptotic expressions for $u \gg \tau$ have the form

$$J_B^{as}(\tau, u, \lambda) \approx \frac{L}{4}\sqrt{\frac{\lambda}{\pi u}} e^{-(1-\lambda)u - \frac{\lambda\tau^2}{4u}}, \tag{18}$$

$$H_B^{as}(\tau, u, \lambda) \approx \frac{L\tau}{4u}\sqrt{\frac{\lambda}{\pi u}} e^{-(1-\lambda)u - \frac{\lambda\tau^2}{4u}}. \tag{19}$$

When $\lambda = 1$, these expressions also coincide with the expressions (7) and (8) of the diffusion approximation.

First of all, let us consider the case A. When $\lambda = 1$, the exact and approximate values of $J(\tau, u, \lambda)$ and $H(\tau, u, \lambda)$ are pretty close to each other, and the asymptotic expressions for these quantities coincide with the expressions for $J_D(\tau, u, 1)$ and $H_D(\tau, u, 1)$ in the diffusion approximation. The ratios J_A^{as}/J_D and H_A^{as}/H_D are shown in Table 1.

A different situation occurs when $\lambda < 1$. A comparison of the exact values $J_A(\tau, u, \lambda)$ and $H_A(\tau, u, \lambda)$ with the approximate values of these quantities shows that they can significantly differ from each other. The asymptotic expressions differs from the corresponding expressions in the diffusion approximation. Their ratio is equal to

$$\frac{J_A^{as}(\tau, u, \lambda)}{J_D(\tau, u, \lambda)} = \frac{H_A^{as}(\tau, u, \lambda)}{H_D(\tau, u, \lambda)} \approx \frac{1}{\sqrt{\lambda}}\ e^{-\frac{\tau^2}{4u}\left(\frac{1}{\lambda}-1\right)}. \tag{20}$$

Since λ is included in the exponent, these ratios may differ significantly from the unity.

Let us consider now the case B. We note that due to the finite speed of light $J(\tau, u, \lambda) = 0$ and $H(\tau, u, \lambda) = 0$ if $u < \tau$ but in the diffusion approximation

Table 1: Ratios of J_A^{as}/J_D and H_A^{as}/H_D for $\lambda = 1$

	$\tau = 1$		$\tau = 10$	
u	J_A^{as}/J_D	H_A^{as}/H_D	J_A^{as}/J_D	H_A^{as}/H_D
I	0.801	0.98	5.30×10^7	8.57×10^6
2	0.916	1.57	7.78×10^2	2.24×10^2
3	0.977	1.76	2.92×10^1	1.15×10^1
4	0.995	1.77	6.92	3.33
5	1.018	1.69	3.27	1.82
6	1.022	1.59	2.12	1.32
7	1.024	1.50	1.63	1.104
8	1.023	1.43	1.38	0.997
9	1.022	1.36	1.23	0.942
10	1.021	1.32	1.14	0.914
15	1.015	1.19	0.993	0.906
20	1.012	1.13	0.971	0.939
30	1.008	1.083	0.977	0.983
40	1.006	1.060	0.985	1.001
50	1.005	1.047	0.990	1.009
60	1.004	1.039	0.993	1.012
80	1.003	1.029	0.996	1.014
100	1.002	1.023	0.998	1.014

$J_D(\tau, u, \lambda) \neq 0$ and $H_D(\tau, u, \lambda) \neq 0$ under this condition as the finite speed of light is not taken into account in this approximation. A comparison of the exact and asymptotic expressions gives approximately the same results, as in the case of A, i.e. $J_B^{as}(\tau, u, 1) = J_D(\tau, u, 1)$ and $H_B^{as}(\tau, u, 1) = H_D(\tau, u, 1)$, but when $\lambda < 1$, $J_B^{as}(\tau, u, \lambda)$ and $H_B^{as}(\tau, u, \lambda)$ are significantly different from $J_D(\tau, u, \lambda)$ and $H_D(\tau, u, \lambda)$, as

$$\frac{J_B^{as}(\tau, u, \lambda)}{J_D(\tau, u, \lambda)} = \frac{H_B^{as}(\tau, u, \lambda)}{H_D(\tau, u, \lambda)} \approx \sqrt{\lambda}\, e^{\frac{\tau^2}{4u}(1-\lambda)}, \tag{21}$$

i.e., these ratios depend strongly on λ (see Tables 2 and 3).

From the above it follows that the replacement of the non-stationary radiation transfer equation by the heat conductive equation should give satisfactory results when $\lambda \approx 1$ and can lead to significant errors when $\lambda < 1$.

This conclusion is also valid in the cases of non-stationary radiative transfer in infinite three-dimensional media illuminated by planar or point sources. Let us give the expressions of the mean intensity and radiation flux in these cases (if one uses the Eddington approximation).

Let us consider an infinite medium illuminated by a momentary isotropic planar source which can be represented in the form of multiple isotropic point sources of luminosity L uniformly distributed on the plane $\tau = 0$ with a surface

Table 2: Values of $J_A(\tau,u)$, $J_D(\tau,u)$, $J_B(\tau,u)$ for $\lambda = 0.5$

	$\tau=1$			$\tau=10$		
u	$J_A(\tau,u)$	$J_D(\tau,u)$	$J_B(\tau,u)$	$J_A(\tau,u)$	$J_D(\tau,u)$	$J_B(\tau,u)$
I	5.50×10^{-2}	6.66×10^{-2}	3.32×10^{-2}	2.84×10^{-5}	1.19×10^{-12}	0
2	3.24×10^{-2}	3.24×10^{-2}	1.82×10^{-2}	3.82×10^{-5}	1.37×10^{-7}	0
3	1.88×10^{-2}	1.67×10^{-2}	1.01×10^{-2}	4.11×10^{-5}	4.37×10^{-6}	0
4	1.09×10^{-2}	8.96×10^{-3}	5.63×10^{-3}	3.91×10^{-5}	1.84×10^{-5}	0
5	6.30×10^{-3}	4.92×10^{-3}	3.18×10^{-3}	3.43×10^{-5}	3.49×10^{-5}	0
6	3.64×10^{-3}	2.75×10^{-3}	1.82×10^{-3}	2.84×10^{-5}	4.44×10^{-5}	0
7	2.11×10^{-3}	1.55×10^{-3}	1.04×10^{-3}	2.25×10^{-5}	4.53×10^{-5}	0
8	1.22×10^{-3}	8.85×10^{-4}	5.94×10^{-4}	1.73×10^{-5}	4.01×10^{-5}	0
9	7.08×10^{-4}	5.08×10^{-4}	3.43×10^{-4}	1.29×10^{-5}	3.25×10^{-5}	0
10	4.11×10^{-4}	2.93×10^{-4}	1.99×10^{-4}	9.37×10^{-6}	2.47×10^{-5}	7.78×10^{-5}
15	2.82×10^{-5}	1.98×10^{-5}	1.36×10^{-5}	1.51×10^{-6}	3.80×10^{-6}	6.95×10^{-6}
20	2.02×10^{-6}	1.41×10^{-6}	9.81×10^{-7}	1.90×10^{-7}	4.10×10^{-7}	5.79×10^{-7}
30	1.11×10^{-8}	7.81×10^{-9}	5.46×10^{-9}	2.09×10^{-9}	3.42×10^{-9}	3.78×10^{-9}
40	6.50×10^{-11}	4.57×10^{-11}	3.20×10^{-11}	1.81×10^{-11}	2.46×10^{-11}	2.41×10^{-11}
50	3.92×10^{-13}	2.76×10^{-13}	1.93×10^{-13}	1.40×10^{-13}	1.68×10^{-13}	1.54×10^{-13}
60	2.41×10^{-15}	1.70×10^{-15}	1.19×10^{-15}	1.02×10^{-15}	1.12×10^{-15}	9.82×10^{-16}
80	9.47×10^{-20}	6.68×10^{-20}	4.70×10^{-20}	4.99×10^{-20}	4.90×10^{-20}	4.06×10^{-20}
100	3.85×10^{-24}	2.71×10^{-24}	1.91×10^{-24}	2.32×10^{-24}	2.12×10^{-24}	1.70×10^{-24}

density of l and flashing at the initial moment of time ($u = 0$). Then, using the diffusion approximation, we have

$$J_D(\tau,u) = \frac{lL}{8\pi\sqrt{\pi}}\,\frac{\sqrt{3-x_1}}{\sqrt{u}}\,\exp\left(-\frac{(3-x_1)\,\tau^2}{4u}\right), \tag{22}$$

$$H_D(\tau,u) = \frac{lL}{4\sqrt{\pi}}\,\frac{\sqrt{3-x_1}}{u\sqrt{u}}\,|\tau|\,\exp\left(-\frac{(3-x_1)\,\tau^2}{4u}\right), \tag{23}$$

when $\tau \gg 1$, $1-\lambda \ll 1$, $u > \sqrt{3-x_1}\beta_2\tau$.

In the case of an infinite medium illuminated by a momentary point source of luminosity L we have

$$J_D(\tau,u) = \frac{L\alpha^2}{32\pi^2\sqrt{\pi}}\,\frac{(3-x_1)^{\frac{3}{2}}}{u\sqrt{u}}\exp\left(-\frac{(3-x_1)\,\tau^2}{4u}\right), \tag{24}$$

$$H_D(\tau,u) = \frac{L\alpha^2}{16\pi\sqrt{\pi}}\,\frac{(3-x_1)^{\frac{3}{2}}}{u^2\sqrt{u}}\,\tau\,\exp\left(-\frac{(3-x_1)\,\tau^2}{4u}\right), \tag{25}$$

when $\tau \gg 1$, $1-\lambda \ll 1$, $u > \sqrt{3-x_1}\beta_2\tau$.

Table 3: Values of $H_A(\tau, u)$, $H_D(\tau, u)$, $H_B(\tau, u)$ for $\lambda = 0.5$

	$\tau = 1$			$\tau = 10$		
u	$H_A(\tau, u)$	$H_D(\tau, u)$	$H_B(\tau, u)$	$H_A(\tau, u)$	$H_D(\tau, u)$	$H_B(\tau, u)$
I	8.69×10^{-2}	6.66×10^{-2}	7.38×10^{-3}	4.96×10^{-5}	1.19×10^{-11}	0
2	4.12×10^{-2}	1.62×10^{-2}	3.57×10^{-3}	6.18×10^{-5}	6.84×10^{-7}	0
3	1.96×10^{-2}	5.57×10^{-3}	1.75×10^{-3}	6.26×10^{-5}	1.46×10^{-5}	0
4	9.38×10^{-3}	2.24×10^{-3}	8.73×10^{-4}	5.64×10^{-5}	4.61×10^{-5}	0
5	4.52×10^{-3}	9.85×10^{-4}	4.41×10^{-4}	4.71×10^{-5}	6.98×10^{-5}	0
6	2.20×10^{-3}	4.58×10^{-4}	2.26×10^{-4}	3.73×10^{-5}	7.41×10^{-5}	0
7	1.08×10^{-3}	2.22×10^{-4}	1.17×10^{-4}	2.83×10^{-5}	6.47×10^{-5}	0
8	5.37×10^{-4}	1.11×10^{-4}	6.12×10^{-5}	2.08×10^{-5}	5.02×10^{-5}	0
9	2.69×10^{-4}	5.64×10^{-5}	3.24×10^{-5}	1.49×10^{-5}	3.61×10^{-5}	0
10	1.37×10^{-4}	2.93×10^{-5}	1.73×10^{-5}	1.04×10^{-5}	2.47×10^{-5}	8.64×10^{-5}
15	5.37×10^{-6}	1.32×10^{-6}	8.39×10^{-7}	1.41×10^{-6}	2.54×10^{-6}	4.78×10^{-6}
20	2.60×10^{-7}	7.07×10^{-8}	4.63×10^{-8}	1.52×10^{-7}	2.05×10^{-7}	2.91×10^{-7}
30	8.66×10^{-10}	2.60×10^{-10}	1.75×10^{-10}	1.27×10^{-9}	1.14×10^{-9}	1.25×10^{-9}
40	3.63×10^{-12}	1.14×10^{-12}	7.79×10^{-13}	8.74×10^{-12}	6.15×10^{-12}	5.60×10^{-12}
50	1.71×10^{-14}	5.51×10^{-15}	3.79×10^{-15}	5.57×10^{-14}	3.36×10^{-14}	3.04×10^{-14}
60	8.62×10^{-17}	2.83×10^{-17}	1.95×10^{-17}	3.43×10^{-16}	1.87×10^{-16}	1.62×10^{-19}
80	2.50×10^{-21}	8.35×10^{-22}	5.80×10^{-22}	1.27×10^{-20}	6.13×10^{-21}	5.04×10^{-21}
100	8.02×10^{-26}	2.71×10^{-26}	1.89×10^{-26}	4.71×10^{-25}	2.12×10^{-25}	1.63×10^{-25}

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Some New Directions of Development of the Radiative Transfer Theory

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It is shown that the problems of radiation transfer in homogeneous plane-parallel atmospheres admit a variational formulation, the equation of transfer then being the Euler–Lagrange equation and the known quadratic and bilinear relations being the conservation law due to form-invariance of the suitable Lagrangian. A group of transfer problems is revealed which are reducible to the source-free problem. We present a group-theoretical description of radiation transfer in inhomogeneous and multi-component atmospheres with plane-parallel geometry. The concept of composition groups is introduced for the media with different optical and physical properties. The group representations are derived for two possible cases of illumination of a composite finite atmosphere from outside. An algorithm for determining the global optical characteristics (reflectance and transmittance) of inhomogeneous and multi-component atmospheres is given. The group theory approach is also applied to determine the field of radiation inside the inhomogeneous atmosphere. The concept of a group of optical depth translations is introduced. The developed theory is illustrated with the problem of radiation diffusion with partial frequency distribution for the case where the inhomogeneity of the medium is due to the depth-variation of the scattering coefficient. It is shown that once reflectance and transmittance of a medium is determined, the internal field of radiation in the source-free atmosphere is found without solving any new equations.

1 Introduction

The research on the theory of radiative transfer carried out in recent two decades in Byurakan observatory develops Ambartsumian's ideas concerning the laws of addition of layers [1, 2] and the principle of invariance [2, 3, 4]. Being of importance for analytical theory itself, new results allow elaborating efficient computational schemes for various astrophysical applications involving radiation transfer in inhomogeneous absorbing and scattering atmospheres. In this context there is a need to define their place and importance in the modern transfer theory.

The report considers results obtained in two directions, the first of which concerns the variational formulation of radiation transfer problems in a plane-parallel homogeneous atmosphere.

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2 Lagrangian formalism

Before turning to immediate description of the variational or Lagrangian approach to radiative transfer problems we will briefly dwell on premises of this research. The fact is that although Ambartsumian's principle of invariance has been known for a long time, but its physical meaning remained obscure. In particular, it was unclear what are the limits of applicability and efficiency of the principle. The second point concerns Rybicki's work [5], where some quadratic integrals of the transfer equation were derived referred by him to as Q- and R-integrals. He supposed that these integrals are possibly related with the principle of invariance. In some problems they lead to non-linear relations linking to each other some characteristics of the radiation field in the atmosphere. Further generalization of Rybicki's results for monochromatic and isotropic scattering in a plane-parallel medium was given in [6, 7], where new sorts of relations were obtained referred to as bilinear and two-point bilinear relations, which couple the radiation fields at different depths of a given atmosphere as well as the radiation fields in different atmospheres.

In frameworks of variational formalism we developed the equations of transfer are proved to be none the other than the Euler-Lagrange equations and the non-linear Q-relations are the conservation laws due to form-invariance of the suitable Lagrangian. In fact, a single functional comprises all the information on features of the problem and allows a systematic connection between symmetries and conservation laws. Being the first integrals of the Euler–Lagrange equation, this laws may facilitate the solution of the problem under consideration and contribute to its interpretation.

To demonstrate the approach, we write the transfer equations in terms of the function Y having the following probabilistic meaning: it characterizes the probability of the photon exit from atmosphere in the direction μ, if originally it was moving at depth τ with the directional cosine η.

We have

$$\pm\frac{dY(\tau,\pm\eta,\mu)}{d\tau} = -Y(\tau,\pm\eta,\mu) + \frac{\lambda}{2}\int_{-1}^{1} Y(\tau,\pm\eta',\mu)d\eta', \tag{1}$$

where λ is the scattering coefficient. The Lagrangian density L corresponding to Eq. (1) was obtained in [8]

$$L(\Phi,\Phi',\tau,\eta,\mu) = \Phi^2 + (\eta\Phi')^2 - 2\Phi U, \tag{2}$$

where we introduced notations

$$\Phi(\tau,\eta,\mu) = Y(\tau,\eta,\mu) + Y(\tau,-\eta,\mu), \qquad U(\tau,\mu) = \frac{\lambda}{2}\int_0^1 \Phi(\tau,\eta',\mu)d\eta'. \tag{3}$$

In accordance with the results of [8], the Euler–Lagrange equation has a form

$$\frac{\partial L}{\partial \Phi} - \frac{d}{d\tau}\frac{\partial L}{\partial \Phi'} + \lambda\int_0^1 \frac{\partial L}{\partial U}d\eta' = 0. \tag{4}$$

One will make sure that insertion of the Lagrangian (2) into Eq. (4) yields the transfer equation (1). It is important that both the transfer equation (1) and the Lagrangian density (2) do not depend explicitly on τ, or stated differently, they are form-invariant under infinitesimal transformation of the optical depth.

This implies that the transformation (or translation) of the optical depth is the symmetry transformation for the system (1). The derivation of conservation laws from direct study of the variational integral is based on Noether's theorem (see, for instance, [9]), which was generalized in [10] to encompass the integro-differential equations. For the problem under consideration, it suggests a conservation law as follows:

$$\int_0^1 \left[L - \frac{\partial L}{\partial \Phi} \Phi' \right] d\eta = const, \tag{5}$$

which, in view of Eq. (2), takes a form

$$\int_0^1 Y(\tau, \zeta, \mu) Y(\tau, -\zeta, \mu) d\zeta = \frac{\lambda}{4} \left(\int_{-1}^1 Y(\tau, \zeta, \mu) d\zeta \right)^2 + const. \tag{6}$$

This relation is, in essence, a prototype of the Q-integral obtained by Rybicki [5]. The above considerations imply that by its content the integral (6) is an analog of the momentum conservation law in mechanics and is due to the axes translation transformation. It holds everywhere where λ does not vary with depth.

The variational formalism allows one not only to elucidate the physical meaning of invariance principle but enables to derive along with many known results a great number of new relations of great importance for the theory and applications. It allows one also to find out some statistical characteristics of the diffusion process in the atmosphere [7, 11]. Some of the known non-linear relations possess a fairly obvious physical or/and probabilistic meaning and can be written immediately on the base of simple arguments.

This approach reveals a group of common radiation transfer problems of astrophysical interest which admit quadratic and bilinear integrals. All of them can be reduced to the source-free problem. This group of problems referred to as RSF-problems includes Milne's problem, the problem of diffuse reflection (and transmission in the case of the atmosphere of finite optical thickness) as well as problems with exponential and polynomial laws for the distribution of internal energy sources. The group problems are characterized at least by three features. First of all, the invariance principle implies bilinear relations connecting the solutions of the listed problems. It was shown in [12] that the group of the RSF-problems admits a class of integrals involving quadratic and bilinear moments of the intensity of arbitrarily high orders. Secondly, if the problem can be formulated for finite atmosphere then the principle allows connecting its solution with that of the proper problem for a semi-infinite atmosphere. Finally, knowledge of the

Ambartsumian φ-function reduces their solutions to the Volterra-type equations for the source function with the kernel-function

$$L(\tau) = \frac{\lambda}{2} \int_0^1 \varphi(\zeta) e^{-\frac{\tau}{\zeta}} \frac{d\varphi}{\zeta}. \tag{7}$$

While the variational approach is widely used in various branches of theoretical physics, it was not the case in the field of the radiative transfer theory, with the only exception being the paper of Anderson [8] who established the conservation law suitable for the case of non-isotropic scattering. We used the results of the rigorous mathematical theory in applying the Lagrangian formalism to the one-dimensional transfer problem [13].

3 Group-theoretical description of radiative transfer in inhomogeneous atmospheres

The next topic of the report concerns application of the group theory to solve the radiative transfer problems in inhomogeneous atmospheres under general assumptions on the frequency-angle distribution of the radiation field, the elementary event of scattering and properties of the medium. As we shall see, the theory we put forward can be regarded as a further extension of the layers adding method proposed first by Ambartsumian [1, 2] for one-dimensional homogeneous media and generalized by Nikoghossian [14, 15] over the case of inhomogeneous media. We remind that the method establishes summation laws for global optical properties of absorbing and scattering media (reflectance and transmittance), which express these properties of the combined medium through similar properties of its components. Of special interest is the particular limiting case of this method when optical thickness of one of the added components tends to zero. This allows one to find the global optical characteristics of a medium simultaneously for a family of the media of different thicknesses. This branch of the theory was developed by Bellman and his co-authors (see, e.g., [16, 17]) and is known as “invariant imbedding”.

3.1 Composition groups

We start with considering the amalgamation procedure of the plane-parallel absorbing and scattering inhomogeneous media. It is assumed that the added components do not contain primary energy sources and are allowed to differ one from the other not only by optical thicknesses, but also by the nature of inhomogeneity. By inhomogeneity we mean that each of the physical parameters specifying the elementary event of scattering or physical state of the medium may vary with depth. Of them we note the profile of the absorption coefficient, the quantum scattering (or destruction) coefficient, Voigt’s parameter, the phase function, the frequency redistribution function, the Stokes parameters in the case of polarized radiation, the correlation length for turbulent media, and so on. However, in illustrating the approach, we restrict ourselves by treating the 1D transfer problem for the case of partial redistribution over frequencies by assuming that the only variable parameter is the scattering coefficient.

Now we introduce the concept of composition or transformation of scattering and absorbing inhomogeneous media, which refers to the addition of a new medium to the initial one. The transformations induced in this way form a group if under the group product (binary operation) one takes the resultant of two successive transformations. It is remarkable that this definition does not specify the nature of inhomogeneity of added media. It is easily seen that all the required conditions for forming a group are satisfied. In particular, the role of the unit element is played by the identity transformation, which leaves the initial medium unchanged, and the inverse element is the transformation which reverses the effect of the already performed transformation. The associativity of the group product is obvious. We refer to this group of transformations as the GN(2,C) group, which, evidently, is not commutative. As a result of the described compositions, one can construct different atmospheres composed of inhomogeneous components.

Of special interest is one of subgroups of the introduced group which describes the case when the added media are homogeneous. The components of such a composite atmosphere may differ from each other not only by optical thicknesses but also by any characteristics of the radiation diffusion in them. Such groups, referred nominally to as GNH(2,C), are two-, three- and multi-parameter dependent on the number of parameters changing in passing from one component to another. The groups of these types are infinite and non-commutative. They can serve as archetypes for a number of real radiating media of astrophysical importance. Finally, of independent interest is the narrower subgroup of the introduced two groups which involves compositions of homogeneous media with identical physical properties but, in general, of different optical thicknesses. These compositions obviously yield homogeneous medium. This one-parameter group, we call it GH(2,C), is infinite and commutative, i.e., Abelian [18]. It becomes continuous when the only parameter, optical thickness, varies continuously.

3.2 The group representations

In order to find the representations of introduced groups, consider a composite atmosphere consisted of two layers, which generally differ in both the optical thickness and functional behavior of parameters specifying the elementary event of scattering (Fig. 1). This means that both components are inhomogeneous and possess the property of polarity [14]. The scattering in the media is supposed occurring with redistribution over directions and frequencies so that the optical characteristics of media may be presented in the operator-matrix form with the matrix elements possessing probabilistic meaning (throughout the paper we use the probability language). They describe the angle and/or frequency dependent probabilities of a single event of reflection and transmission. Having in mind

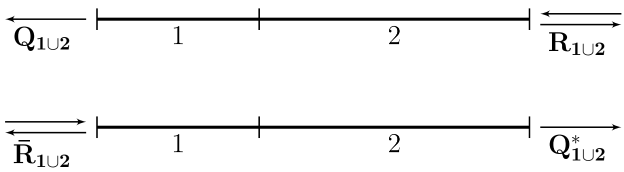

Figure 1: Reflection and transmission by inhomogeneous atmosphere.

the polarity property of inhomogeneous media, we introduce the notations $\mathbf{R}_i$, $\mathbf{Q}_i$ and $\bar{\mathbf{R}}_i$, $\bar{\mathbf{Q}}_i$ $(i = 1, 2)$ for the reflection and transmission coefficients of the components of a composite medium illuminated correspondingly from the right and left. In accordance with the principle of reversibility of optical phenomena, $\bar{\mathbf{Q}}_i = \mathbf{Q}_i^*$, where the transposed matrix is supplied by asterisk. Everywhere below we follow the designation $\mathbf{Q}_i^*$. An important role in this research belongs to the inverse of the transmittance matrix $\mathbf{P} = \mathbf{Q}^{-1}$ and the other three combined matrices $\mathbf{S} = \mathbf{R}\mathbf{P}$, $\bar{\mathbf{S}} = \mathbf{P}\bar{\mathbf{R}}$, $\mathbf{M} = \mathbf{Q}^* - \mathbf{S}\bar{\mathbf{R}}$. These four matrices provide a complete description of the optical properties of an inhomogeneous absorbing and scattering medium independent of that what of its boundaries is illuminated from outside.

Let us treat now the transfer of radiation through composite medium when a photon falls on its right boundary (top drawing in Fig. 1). Taking account of possibility of multiple reflections between components of the medium, one can derive the following two relations (see [19]):

$$\mathbf{P}_{1\cup 2} = \mathbf{P}_2\mathbf{P}_1 - \bar{\mathbf{S}}_2\mathbf{S}_1, \tag{8}$$

$$\mathbf{S}_{1\cup 2} = \mathbf{S}_2\mathbf{P}_1 + \mathbf{M}_2\mathbf{S}_1, \tag{9}$$

where the quantities pertaining to composite medium are indexing with $1 \cup 2$.

Taking together, relations (8) and (9) can be presented in the more convenient compact form

$$\begin{pmatrix} \mathbf{P}_{1\cup 2} \\ \mathbf{S}_{1\cup 2} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_2 & -\bar{\mathbf{S}}_2 \\ \mathbf{S}_2 & \mathbf{M}_2 \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{S}_1 \end{pmatrix}, \tag{10}$$

where we used the concepts of supervector and supermatrix [18, 20, 21]. The supermatrix entering in Eq. (10) is denoted by $\tilde{\mathbf{A}}$ (hereafter the supermatrices are supplied by tilde)

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{P} & -\bar{\mathbf{S}} \\ \mathbf{S} & \mathbf{M} \end{pmatrix}. \tag{11}$$

The set of matrices $\tilde{\mathbf{A}}$ is the first of representations of the group of compositions GN(2,C) which also is a group (we denote it by g) and provides a one-to-one mapping of GN(2,C) to supervector space, i.e., the group product of two transformations $g_1 \bigotimes g_2$, corresponds to $\tilde{\mathbf{A}}_{1\cup 2} = \tilde{\mathbf{A}}_1\tilde{\mathbf{A}}_2$, or for representations $\Im(g_1 \bigotimes g_2) = \Im(g_1)\Im(g_2)$ (isomorphism). On the hand, the supermatrix $\tilde{\mathbf{A}}$ can be regarded as an operator mapping one supervector space to another one.

It is natural to refer nominally to this supermatrix as “composer”. It plays an important role in the developed theory.

It is easy to see that the transformation realizing by $\tilde{\mathbf{A}}$ provides determination of optical properties of the composed medium partially, namely, only those for the right-hand side illumination. For complete description of optical properties of the composite medium, we need the matrices $\bar{\mathbf{S}}$ and $\mathbf{M}$ which obey the following transformations [19]:

$$\bar{\mathbf{S}}_{1\cup 2} = \mathbf{P}_2\bar{\mathbf{S}}_1 + \bar{\mathbf{S}}_2\mathbf{M}_1, \qquad \mathbf{M}_{1\cup 2} = \mathbf{M}_2\mathbf{M}_1 - \mathbf{S}_2\bar{\mathbf{S}}_1. \tag{12}$$

Note that these relations could be derived directly.

In the matrix-operator form they read

$$\begin{pmatrix} \mathbf{M}_{1\cup 2} \\ \mathbf{S}_{1\cup 2} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_2 & -\mathbf{S}_2 \\ \bar{\mathbf{S}}_2 & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} \mathbf{M}_1 \\ \bar{\mathbf{S}}_1 \end{pmatrix}. \tag{13}$$

Thus, we are led to an alternative group of representations given by the supermatrix

$$\tilde{\mathbf{B}} = \begin{pmatrix} \mathbf{M} & -\mathbf{S} \\ \bar{\mathbf{S}} & \mathbf{P} \end{pmatrix}, \tag{14}$$

which we denote by $\bar{\Im}(g)$. It is evident that this group also is isomorphic to the group of compositions GN(2,C) and together with $\Im(g)$ gives a complete description of optical properties of the composite atmosphere illuminated from the right. In both cases the identity transformation is given by the supermatrix

$$\tilde{\mathbf{E}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{15}$$

where $\mathbf{I}$ is the unit matrix. The supermatrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ are non-degenerate, and two-sided inverse matrices exist with superdeterminant [21, 22, 23] equaled to one (see [19]).

By introducing the four-dimensional supervector $\tilde{\mathbf{Y}}$ with the components $(\mathbf{P}, \mathbf{S}, \mathbf{M}, \bar{\mathbf{S}})$, the group representations $\Im(g)$, $\bar{\Im}(g)$ can be joined and presented as a reducible representation

$$\tilde{\mathbf{Y}}_{1\cup 2} = \tilde{\mathbf{\Psi}}_2\tilde{\mathbf{Y}}_1, \tag{16}$$

where

$$\tilde{\mathbf{\Psi}} = \begin{pmatrix} \mathbf{P} & -\bar{\mathbf{S}} & \mathbf{0} & \mathbf{0} \\ \mathbf{S} & \mathrm{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{M} & -\mathbf{S} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{S}} & \mathbf{P} \end{pmatrix}. \tag{17}$$

We conclude that, given the optical properties of the component layers, the common matrix multiplications allow one to determine these properties for the compound atmosphere. If the atmosphere is homogeneous one can restrict oneself by transformation Eq. (10). Arguments analogous to those above in deriving Eq. (17) allow one to derive adding laws for the case when the composite atmosphere is illuminated from the side of the left boundary (bottom drawing in Fig. 1) [19].

3.3 The 1D source-free problem for partial redistribution over frequencies

Consider a subgroup of the composition group GNH(2,C) subjected to the only condition that the optical thickness of the medium obtained as a result of compositions must not exceed some presetting value of τ_0. When the optical thickness varies continuously, this infinite group is obviously continuous. Then this group together with its representation $\Im(g)$ are one-dimensional Lie groups [21, 22, 23]. With help of compositions of this groups one can construct a multi-component atmosphere with components which generally can differ one from the other by their physical characteristics.

As an example, let us treat the matrix problem of radiation diffusion in a one-dimensional inhomogeneous atmosphere illuminated from the boundary $\tau = \tau_0$ when the scattering obeys the angle averaged law of partial redistribution over frequencies. Suppose that the atmosphere consists of components of equal and sufficiently small thickness characterized by some constant values of the scattering coefficient λ, so that in the limit of the components thicknesses tending to zero it might be regarded as a continuous function of the optical depth.

The infinitesimal operator of this group of compositions at τ_0 can be represented in the form

$$\tilde{\mathbf{\Xi}}(\tau_0) = \lim_{\Delta\tau_0 \to 0} \frac{\widetilde{\mathbf{A}}(\tau_0 + \Delta\tau_0) - \widetilde{\mathbf{A}}(\tau_0)}{\Delta\tau_0} = \begin{pmatrix} \mathbf{m}(\tau_0) & -\mathbf{n}(\tau_0) \\ \mathbf{n}(\tau_0) & -\mathbf{m}(\tau_0) \end{pmatrix}, \tag{18}$$

where

$$\mathbf{m}(\tau_0) = \alpha - \mathbf{n}(\tau_0), \qquad \mathbf{n}(\tau_0) = \frac{\lambda(\tau_0)}{2}\mathbf{\Gamma}. \tag{19}$$

Here α and $\mathbf{\Gamma}$ are the discrete analogs correspondingly of the profile of the absorption coefficient and the law of the frequency redistribution [24]. For the sake of simplicity, they are supposed to be independent of depth. Evidently, $\mathbf{\Gamma}$ is a symmetric matrix and α is a diagonal matrix with the elements $\alpha_i = \alpha(x_i)$.

Transformation (8) implies [25]

$$\frac{d\mathbf{P}}{d\tau_0} = \mathbf{m}(\tau_0)\,\mathbf{P}(\tau_0) - \mathbf{n}(\tau_0)\,\mathbf{S}(\tau_0), \tag{20}$$

$$\frac{d\mathbf{S}}{d\tau_0} = \mathbf{n}(\tau_0)\,\mathbf{P}(\tau_0) - \mathbf{m}(\tau_0)\,\mathbf{S}(\tau_0), \tag{21}$$

with the initial conditions $\mathbf{P}(0) = \mathbf{I}$, $\mathbf{S}(0) = \mathbf{0}$, where $\mathbf{0}$ is the null matrix.

Inversion of the matrix $\mathbf{P}(\tau_0)$ found from the set of equations (20) and (21) allows one to determine the requisite values of the medium reflectance and transmittance. Analogously, by using the infinitesimal operator of the supermatrix $\tilde{\mathbf{B}}$ and Eq. (14), we are led to a new set of the matrix differential equations

$$\frac{d\mathbf{M}}{d\tau_0} = -\mathbf{m}(\tau_0)\,\mathbf{M}(\tau_0) - \mathbf{n}(\tau_0)\,\bar{\mathbf{S}}(\tau_0), \tag{22}$$

$$\frac{d\bar{\mathbf{S}}}{d\tau_0} = \mathbf{n}(\tau_0)\,\mathbf{M}(\tau_0) + \mathbf{m}(\tau_0)\,\bar{\mathbf{S}}(\tau_0), \tag{23}$$

with the initial conditions $\mathbf{M}(0) = \mathbf{I}$, $\bar{\mathbf{S}}(0) = \mathbf{0}$.

In the case of homogeneous atmosphere one can restrict oneself to solving the set of equations (20)–(21). Its solution can be presented in the form of the matrix exponential [25]. Note that from the sets of equations (20)–(23) one can derive separate matrix differential equations of the second order for unknown matrix-functions as it is the case in the scalar case [25].

Equations obtained with the group approach exhibit intimate connection between the group approach and the method of invariant imbedding [16, 17]. As a matter of fact, the invariant imbedding technique is equivalent to action of infinitesimal operators of the proper group representations introduced in the paper. For homogeneous atmosphere, the obtained equations admit invariants or conservation laws, the continual analogs of which were obtained in the mentioned papers [7, 8, 25, 26].

The efficiency of the developed theory becomes especially discernible when solving radiative transfer problems for atmospheres with a complex multi-layer structure. In applying any of the introduced composers, one needs to predetermine the global optical properties of each of the layers added to the boundary $\tau = \tau_0$, namely, the matrices $\mathbf{P}$, $\mathbf{S} = \mathbf{R}\mathbf{P}$, $\bar{\mathbf{S}} = \mathbf{P}\bar{\mathbf{R}}$ and $\mathbf{M} = \mathbf{Q}^* - \mathbf{S}\bar{\mathbf{R}} = \mathbf{Q}^* - \mathbf{R}\bar{\mathbf{S}}$, i.e., the triad of matrices $\mathbf{R}$, $\bar{\mathbf{R}}$, $\mathbf{Q}$. The problem is simpler when the components are homogeneous. Particularly, in the scalar problems these quantities are determined analytically. In the general case of inhomogeneous components, we can turn to solutions of the systems of equations (20)–(23) with subsequent inversion of the matrix $\mathbf{P}$. This route is preferable in finding the field of radiation inside the medium to be discussed below. However, there exists an alternative way of determining the required optical properties by solving basic differential equations obtained in [12, 27], which are easily realizable initial-value problems.

Thus, the algorithm of solution of the transfer problem in the most general case of multi-component atmosphere is as follows. One starts with finding the reflectance and transmittance of the layers to be added by using one of the routes described above. Further, the compositions transformations are continued until the optical thickness of the composite atmosphere specified by the problem formulation is attained. Inversion of the matrix $\mathbf{P}(\tau_0)$ allows one to find $\mathbf{Q}(\tau_0)$ what, in its turn, determines other properties of the composite atmosphere. We shall see below that the obtained quantities are sufficient to find the field of radiation inside the medium.

In the special case when the supplemented layers are homogeneous and possess similar properties, we deal with the cyclic group and the composition process reduces to the action of powers of corresponding operators ($\tilde{\mathbf{A}}^n$, for instance). This naturally reduces the volume of computations to a great extent.

3.4 Radiation field inside the medium

The goal we pursue in this section is to extend the group theory approach over the field of radiation inside inhomogeneous media. Consider a plane-parallel inhomogeneous atmosphere of optical thickness τ_0, the boundary $\tau = \tau_0$ of which is illuminated from outside (Fig. 2). Light scattering is generally assumed occurring with the angle and frequency redistribution. The internal field of radiation we assign by the matrices $\mathbf{U}(\tau, \tau_0)$ and $\mathbf{V}(\tau, \tau_0)$, which specify the probabilities that the quantum with the angle-frequency characteristics (η, x) falling on the boundary $\tau = \tau_0$, will be found, as a result of diffusion in the medium, at the depth τ moving correspondingly to the boundaries $\tau = 0$, and $\tau = \tau_0$, generally with some other characteristics (η', x').

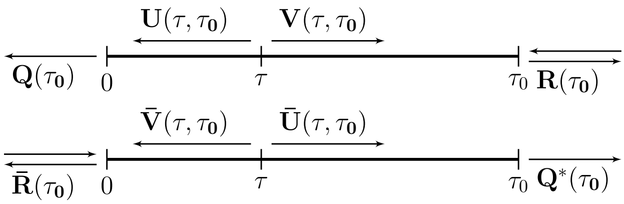

Figure 2: Description of the radiation field inside the inhomogeneous atmosphere.

Let us treat now the procedure of transition from one optical depth to another one by supplementing a new layer. The infinite set of such transitions obviously composes a group if the group product is defined as the result of two subsequent transitions. One can easily check that all the group postulates are satisfied. In accordance with the physics of the problem, the resulting value of the optical depth should not exceed the optical thickness of the medium $\tau \leq \tau_0$. This group is a subgroup of the group GN(2,C) and is equivalent to the similar subgroup considered in the preceding sections for composition of different media.

Taking into account the probability meaning of matrices $\mathbf{U}(\tau, \tau_0)$ and $\mathbf{V}(\tau, \tau_0)$, one can write

$$\mathbf{Q}(\tau_0) = \mathbf{Q}(\tau)\, \mathbf{U}(\tau, \tau_0), \quad \mathbf{V}(\tau, \tau_0) = \mathbf{R}(\tau)\, \mathbf{U}(\tau, \tau_0), \tag{24}$$

hence

$$\mathbf{U}(\tau, \tau_0) = \mathbf{P}(\tau)\, \mathbf{Q}(\tau_0), \quad \mathbf{V}(\tau, \tau_0) = \mathbf{S}(\tau)\, \mathbf{Q}(\tau_0). \tag{25}$$

The fact of separation of arguments in $\mathbf{U}(\tau, \tau_0)$ and $\mathbf{V}(\tau, \tau_0)$ is one of advantages of the applied approach. Equations (24) imply that the subgroup of representation $\Im(g)$ relevant to the media compositions group may be now regarded as representation of the depth-translation group.

Indeed, on the base of Eq. (10), one may write

$$\begin{pmatrix} \mathbf{U}(\tau + \delta\tau, \tau_0) \\ \mathbf{V}(\tau + \delta\tau, \tau_0) \end{pmatrix} = \begin{pmatrix} \mathbf{P}_\tau(\delta\tau) & -\bar{\mathbf{S}}_\tau(\delta\tau) \\ \mathbf{S}_\tau(\delta\tau) & \mathbf{M}_\tau(\delta\tau) \end{pmatrix} \begin{pmatrix} \mathbf{U}(\tau, \tau_0) \\ \mathbf{V}(\tau, \tau_0) \end{pmatrix}, \tag{26}$$

where $\delta\tau$ is an increment to the optical depth τ. The subscript τ indicates that the internal physical properties of supplemented layer are relevant to (or vary in) the interval $(\tau, \tau + \delta\tau)$.

Thus, the supermatrix $\tilde{\mathbf{A}}$ plays an important role not only in adding the media of different optical thicknesses but also in translating optical depths inside inhomogeneous atmosphere. Stating differently, it serves at the same time as "composer" of inhomogeneous atmospheres and as "translator" in transitions between optical depths inside the atmosphere. It is noteworthy that in the latter case only the global optical properties of the incremented layer provide the transformations. The internal physical characteristics do not take an immediate part in these transformations, so that the nature of inhomogeneity in different media or layers are allowed to be different.

To illustrate the obtained results, let us return to the matrix case of the transfer problem treated in Section 3.3, where we confined ourselves to the global optical characteristics of the medium. Our immediate objective now is to find the field of radiation inside the medium, where, again, the only parameter varying with depth is the scattering coefficient λ. In light of that said in Sect. 3.3, we conclude that the depth-translation group together with its representation are the Lie groups of the one-dimension.

Given the supermatrix (18), the transformation (26) leads to the customary differential equations of radiation transfer for the operator-functions $\mathbf{U}$ and $\mathbf{V}$

$$\frac{d\mathbf{U}}{d\tau_0} = \mathbf{m}(\tau)\,\mathbf{U}(\tau,\tau_0) - \mathbf{n}(\tau)\,\mathbf{V}(\tau,\tau_0), \tag{27}$$

$$\frac{d\mathbf{V}}{d\tau_0} = \mathbf{n}(\tau)\,\mathbf{U}(\tau,\tau_0) - \mathbf{m}(\tau)\,\mathbf{V}(\tau,\tau_0). \tag{28}$$

In place of the usual boundary conditions, one can now adopt the conditions at $\tau = \tau_0$, $\mathbf{U}(0,\tau_0) = \mathbf{Q}(\tau_0)$, $\mathbf{V}(0,\tau_0) = \mathbf{0}$, then reducing the problem to that with initial conditions. Derivation of the transfer equations (27)–(28) on the base of physical reasoning is straightforward, what is usually doing in the classical astrophysical literature. As it was shown, the operator-functions $\mathbf{P}(\tau)$ and $\mathbf{S}(\tau)$ satisfy the same set of equations (20)–(21) with the initial conditions $\mathbf{P}(0) = \mathbf{I}$, $\mathbf{S}(0) = \mathbf{0}$. By comparing the initial conditions of these two sets of equations, we are led to relations (24) written above on the base of probabilistic reasoning [26].

Bearing in mind the computations described in Section 3.4 for the composite inhomogeneous atmosphere as well as the equivalence of the medium-composition and the depth-translation subgroups of GNH(2,C), we arrive at an important conclusion that the internal field of radiation now can be found without solving any new equations. Indeed, it is sufficient to this end to multiply the obtained value of $\mathbf{Q}(\tau_0)$ by $\mathbf{P}$ and $\mathbf{S}$ found above in intermediate calculations in constructing the atmosphere under study.

The far reaching analogy between media composition and depths translation groups makes it possible to transfer different results obtained for global optical properties of an atmosphere to quantities determining the internal field of radiation. For instance, if the atmosphere is homogeneous, one can derive conservation laws in terms of **U** and **V**, as it was done above for the matrices **P** and **S**. We do not deal with it here but refer the interested reader after continual analogs of these laws to [7, 26].

4 Conclusions

We discussed two directions of further development of the radiation transfer theory which, in our opinion, are promising from both the analytical and computational points of view. They generalize Ambartsumian's ideas concerning the principle of invariance and the layers adding laws. The variational approach allows one to reveal the physical nature and the scope of applicability of invariance principle. It is important that the solutions of some standard problems of astrophysical interest mathematically are reducible to the Volterra type integral equations.

The second direction concerns the group theory which is applied to solve the problems of radiative transfer in inhomogeneous absorbing and scattering atmospheres. The media composition groups and their representations introduced in the paper generalize the layers adding approach, which now covers inhomogeneous, particularly multi-component, atmospheres with allowance of the angle and frequency distribution of the radiation field. The group representations being expressed in terms of some combined discrete quantities allow one to find the most general summation laws for reflectance and transmittance of the plane-parallel media.

Employment of infinitesimal operators of the introduced groups makes it possible to establish the close connection of the introduced groups with the classical transfer equations and the equations ensuing from invariant imbedding. In fact, the first of them are connected with the depth translation groups, while the second – with composition groups for the media of different optical thicknesses.

An important result in considering the internal field of radiation is the separation of variables of the optical depth and thickness in the expression of quantities describing the optical properties. This implies that the introduced group of the optical depths translations is a subgroup of the group of the media compositions. In its turn, this means that after finding the reflectance and transmittance of an atmosphere, there is no need to solve any new equations to determine the internal field of radiation in the source-free atmosphere.

The theory we put forward is of sufficiently great generality since it does not depend on the nature of inhomogeneity of the media as well as on the angle and frequency distribution of the radiation field.

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On the Linear Properties of the Nonlinear Problem of Radiative Transfer

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We address the nonlinear problem of reflection/transmission of radiation from an anisotropic scattering/absorbing one-dimensional medium of finite geometrical thickness, when both of its boundaries are illuminated by intense monochromatic radiative beams. The new conceptual element of so-called “linear images” is noteworthy, which admits a probabilistic interpretation. The solution of nonlinear problem of reflection/transmission of radiation is reduced to a linear combination of linear images. They describe the reflectivity and transmittance of the medium for a single photon or their beam of unit intensity, incident on one of the boundaries of the layer, when the medium in real regime is still under the bilateral illumination by external exciting radiation of arbitrary intensity. To determine the linear images, we exploit three well known methods: (i) adding of layers, (ii) its limiting form described by differential equations of invariant imbedding, and (iii) a transition to the so-called functional equations of Ambartsumyan's “complete invariance”.

1 Introduction

In linear problem of transfer of radiation energy, the resulting characteristics of the radiation field are formed in the process of multiple interactions of radiation with matter, when the physical properties of the medium are assumed to be unchanged. The very complexity of nonlinear problem, in contrast, is the functional dependence of the scattering/absorbing properties of each elementary volume $\Delta \to 0$ on the intensity of radiation incident on it from all sides. The characteristics of the diffusing in medium radiation field and the physical state of the medium itself are forming each other reciprocally, in a self-consistent manner.

It is well known that in the linear case, the solution of the problem of reflection-transmission (PRT) of radiation, i.e. seeking the intensities $u_L^{\pm}(x,y)$ of emerging radiation from the right “+” and left “−” boundaries of the anisotropic medium (of finite geometrical thickness L), which is illuminated from both boundaries simultaneously by intense radiation beams with intensities x and y, respectively, is reduced to a simple linear combination of the solutions of the two separate problems of its unilateral illumination (from left x, and from right y, separately):

$$u_L^{+}(x,y) = q^{+}\,x + r^{+}\,y, \tag{1}$$

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$$u_L^-(x,y) = r^- x + q^- y, \tag{2}$$

where r^- and q^+ are the coefficients of reflection and transmission of an anisotropic medium of geometric thickness L, for a "single quantum", or their "beam of unit intensity", incident from its left boundary, while r^+ and q^- are their counterparts related to the right boundary. These coefficients can be readily interpreted as the probability densities of reflection and transmission of a single photon incident on one of the two boundaries of medium. In nonlinear case [1]–[5], the relations analogous to Eqs. (1)–(2) do not hold. The relationship of these two problems is now implemented (instead of Eqs. (1)–(2)) by Cauchy problems [1]–[5]. Moreover, it is obvious that in the nonlinear case, it makes no sense at all to operate with such concepts as "single photon", or their "beam of unit intensity", and the use of probabilistic interpretation of transference phenomena, which though are very efficient tools in the linear problems. This obstacle, in nonlinear problems of bilateral illumination of medium, still prevents to explore only the solution of equations for particular PRT of unilateral illumination of medium (such as seeking the variables $r^{\pm}$ and $q^{\pm}$ [6, 7] of the linear case). Therefore, the exact methods of determining the field of radiation emerging from the medium, such as: (i) adding of layers, (ii) its limiting form, described by differential equations of invariant imbedding, and (iii) the so-called functional equations of Ambartsumyan's "complete invariance" (ACI) [4, 5], are compelled here to apply directly to the functions $u_L^{\pm}(x,y)$ of bilateral illumination of medium, which significantly complicates their determination.

A major goal of this report is to simplify further the methods of nonlinear PRT by revealing and exploring some new functions of so-called "linear images" of the solution of PRT. It is noteworthy that the solution in quest of nonlinear PRT is expressed in terms of newly introduced functions explicitly, just as in the linear case, through a simple linear combination of the solutions of more particular PRT of unilateral illumination of medium. We show that the introduction of these linear images allows to handle effectively a random walk of a single quantum or their unit beam. Moreover, this ensures an application of Sobolev's probabilistic interpretation [8] of linear transfer problems, in nonlinear case too, as simple as in the linear case. For a determination of linear images, as a consequence of the systematic application of the principle of invariance [1] and [4]–[6], we explore in unified way the analogues of described above all three methods of solutions of PRT.

2 The linear images of nonlinear PRT

The idea of introduction of linear images is closely related to one observation of Ambartsumian [9, 2] that inevitably a translucence of medium occurs at high intensities of external radiation exiting it, which is due to the transition of essential fraction of atoms from the ground state to an excited. As a result, the proportion of absorbing neutral atoms in the medium decreases and a stationary regime was established in the excited medium with a new, changed, optical thickness. The original problem becomes linear with respect to new values of the optical thickness, unknown in advance. This is just a physical meaning of the approach entitled "method of self-consistent optical depths", and further used very effectively by [10]. Following Ambartsumian, let us trace the path of a single quantum, randomly walking in an anisotropic scattering/absorbing medium, when certain steady state conditions are established in it. This constant level of an excitation of medium is maintained (stationary regime) during the whole process of random walk of a quantum. It means that an arbitrary chosen quantum just "lives" in a linear medium during this entire time. If x number of photons are incident on medium from the left, and y – from the right, then their total output, as in linear case, can be given by the relations analogous to Eqs. (1)–(2), with the only difference that the functions, $R_L^{\pm}(x, y)$ and $T_L^{\pm}(x, y)$, of the described above linear images of solution of original PRT, are dependent on the total number of photons (x, y), because of nonlinearity of the problem:

$$u_L^{+}(x, y) = T_L^{+}(x, y)\, x + R_L^{+}(x, y)\, y, \tag{3}$$

$$u_L^{-}(x, y) = T_L^{-}(x, y)\, y + R_L^{-}(x, y)\, x. \tag{4}$$

The functions $R_L^{\pm}(x, y)$ and $T_L^{\pm}(x, y)$ are the above-mentioned linear images of the solution of original PRT. They are, respectively, the probability densities of reflection and transmission of a single photon or their unit beam incident on the medium through from one of its two boundaries. Although these functions describe the behavior of a single quantum or their unit beam, but because of nonlinearity of the problem nevertheless depend upon the intensities (x, y) of entering medium radiation, due to which the acting level of an excitation of medium has been set. In asymptotic limit of weak fields $x + y \leq \delta^{\pm}$, these functions apparently become constants, which are the solutions of a linear problem, where $\delta^{\pm}$ is the asymptotic threshold of incident single quantum from the left and right, respectively.

3 Relations of the adding of layers for the linear images

As a first method for determining the linear images, let us employ a general method of adding of layers in the nonlinear problems of transfer [1, 2, 5]. Suppose the anisotropic one-dimensional medium of geometrical thickness of B is adjoined from the right to a similar medium of thickness A. Thereby the composite slab of finite thickness $A + B$ is illuminated from the left and right boundaries by radiation of intensities x and y, respectively. Required to determine the intensity of the radiation $u_{A+B}^{\pm}(x, y)$ emerging from this composite slab, when the solutions of similar problems for its both sub-layers, $u_A^{\pm}(x, y)$, $u_B^{\pm}(x, y)$, are previously known. From Eqs. (3)–(4), it is seen that the problem is reduced to determination of the linear images $R^{\pm}_{A+B}(x,y)$ and $T^{\pm}_{A+B}(x,y)$ by means of known linear images $R^{\pm}_{A}(x,y)$, $T^{\pm}_{A}(x,y)$ and $R^{\pm}_{B}(x,y)$, $T^{\pm}_{B}(x,y)$. From the formulas of the nonlinear addition of layers [5], by virtue of Eqs. (3)–(4), we obtain

$$T^{+}_{A+B}(x,y) = T^{+}_{B}\,(p,y)\;p_{+}, \tag{5}$$

$$R^{+}_{A+B}(x,y) = R^{+}_{B}\,(p,y) + T^{+}_{B}\,(p,y)\;p_{-}, \tag{6}$$

$$R^{-}_{A+B}(x,y) = R^{-}_{A}\,(x,s) + T^{-}_{A}\,(x,s)\;s_{+}, \tag{7}$$

$$T^{-}_{A+B}(x,y) = T^{-}_{A}\,(x,s)\;s_{-}, \tag{8}$$

where the four auxiliary functions can be obtained exploring the explicit relations

$$p_{+} = \frac{T^{+}_{A}\,(x,s)}{1 - R^{+}_{A}\,(x,s)\;R^{-}_{B}\,(p,y)}, \quad s_{+} = \frac{R^{-}_{B}\,(p,y)\;T^{+}_{A}\,(x,s)}{1 - R^{-}_{B}\,(p,y)\;R^{+}_{A}\,(x,s)}, \tag{9}$$

$$p_{-} = \frac{R^{+}_{A}\,(x,s)\;T^{-}_{B}\,(p,y)}{1 - R^{+}_{A}\,(x,s)\;R^{-}_{B}\,(p,y)}, \quad s_{-} = \frac{T^{-}_{B}\,(p,y)}{1 - R^{-}_{B}\,(p,y)\;R^{+}_{A}\,(x,s)}. \tag{10}$$

The unknowns, p and s, can be found from the system

$$\begin{cases} p = T^{+}_{A}\,(x,s)\;x + R^{+}_{A}\,(x,s)\;s, \\ s = T^{-}_{B}\,(p,y)\;y + R^{-}_{B}\,(p,y)\;p, \end{cases} \tag{11}$$

or writing them in the form of separate equations

$$p = g^{+}\,(x,y;p,s) + K^{+}\,(x,y;p,s)\;p, \quad s = g^{-}\,(x,y;p,s) + K^{-}\,(x,y;p,s)\;s. \tag{12}$$

Here the proper kernels and free terms are defined by

$$\begin{aligned} K^{+}\,(x,y;p,s) &\equiv R^{+}_{A}\,(x,s)\;R^{-}_{B}\,(p,y)\,, \\ K^{-}\,(x,y;p,s) &\equiv R^{-}_{B}\,(p,y)\;R^{+}_{A}\,(x,s)\,, \\ K^{+}\,(x,y;p,s) &= K^{-}\,(x,y;p,s)\,, \end{aligned} \tag{13}$$

$$g^{+}\,(x,y;p,s) \equiv T^{+}_{A}\,(x,s)\;x + R^{+}_{A}\,(x,s)\;T^{-}_{B}\,(p,y)\;y, \tag{14}$$

$$g^{-}\,(x,y;p,s) \equiv T^{-}_{B}\,(p,y)\;y + R^{-}_{B}\,(p,y)\;T^{+}_{A}\,(x,s)\;x. \tag{15}$$

When one of the equations (12) is already solved, the solution of the other can be obtained directly by using corresponding explicit relation (11). Whereas the attention is drawn to the fact that in the equations (12):

1. The discussed explicit structures have until now met only in linear problems, with the ensuing advantages.
2. Furthermore, an increase of the intensity of external radiation that excites the medium, in the form of direct dependence appears only in free terms g^{+}, g^{-} of these equations. This, as well known, does not affect a convergence of the iterative solutions of considered equations, because it is due only to the properties of kernels.
3. The kernels of equations K^{+}, K^{-} are just the probability densities.

Aforesaid ensures a convergence, for example, of a simple iterative scheme

$$p^{(n+1)} = g^{+}_{(n)} + K^{+}_{(n)}\, p^{(n)} \quad \text{at} \quad s^{(0)} = y, \tag{16}$$

where

$$g^{+}_{(n)} \equiv g^{+}\left(x,\, y;\, p^{(n)},\, s^{(n)}\right), \quad K^{+}_{(n)} \equiv K^{+}\left(x,\, y;\, p^{(n)},\, s^{(n)}\right). \tag{17}$$

In the framework of the method of adding of layers, to determine the linear images of nonlinear PRT, the following sequential scheme can be distinguished: to begin with, we determine p and s from Eqs. (11)–(17), next it will be $p_{\pm}$ and $s_{\pm}$ from Eqs. (9)–(10), afterward $R^{\pm}_{L}(x, y)$, $T^{\pm}_{L}(x, y)$ from Eqs. (5)–(8), and finally $u^{\pm}_{L}(x, y)$ from Eqs. (3)–(4).

4 A complete set of equations of invariant imbedding for the linear images

As a second method for determining the linear images, we derive a complete set of equations of invariant imbedding. More consistent way is to fulfill a limiting transition in the general relations of addition of layers, which were built above, i.e. successively letting one layer be elementary $\Delta \to 0$, while the other is left fixed: $A \equiv \Delta$, $B \equiv L$ and $A \equiv L$, $B \equiv \Delta$. For radiation characteristics of diffuse reflection-transmission of elementary volume can be obtained the explicit forms

$$\begin{aligned} T^{\pm}_{\Delta}(x, y) &= 1 - æ^{\pm}(x, y)\,\Delta + O\left(\Delta^{2}\right), \\ R^{\pm}_{\Delta}(x, y) &= \chi^{\pm}(x, y)\,\Delta + O\left(\Delta^{2}\right). \end{aligned} \tag{18}$$

The physical meaning of the functions $æ^{\pm}(x, y)$ and $\chi^{\pm}(x, y)$ is as follows: they represent the probability densities that the quantum moving in a certain direction will first be absorbed by elementary layer of the medium, and then: a) $æ^{\pm}(x, y)$ will not be re-emitted in the same direction; b) $\chi^{\pm}(x, y)$ will be re-emitted in backwards. Hence a complete set of the equations of invariant imbedding can be written as follows:

$$\left[\frac{\partial}{\partial L} - \hat{E}_{+}\right] T^{+} = -T^{+}\, æ^{+}\left(x, u^{-}\right) + T^{+}\, \chi^{+}\left(x, u^{-}\right)\, R^{-}, \tag{19}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_{+}\right] R^{+} = T^{+}\, \chi^{+}\left(x, u^{-}\right)\, T^{-}, \tag{20}$$

$$\begin{aligned} \left[\frac{\partial}{\partial L} - \hat{E}_{+}\right] R^{-} = {} & \chi^{-}\left(x, u^{-}\right) - R^{-}\, æ^{+}\left(x, u^{-}\right) - \\ & - æ^{-}\left(x, u^{-}\right)\, R^{-} + R^{-}\, \chi^{+}\left(x, u^{-}\right)\, R^{-}, \end{aligned} \tag{21}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_{+}\right] T^{-} = -æ^{-}\left(x, u^{-}\right)\, T^{-} + R^{-}\, \chi^{+}\left(x, u^{-}\right)\, T^{-}, \tag{22}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_{-}\right] T^{+} = -æ^{+}(u^{+}, y)\, T^{+} + R^{+}\, \chi^{-}(u^{+}, y)\, T^{+}, \tag{23}$$

$$\begin{aligned} \left[\frac{\partial}{\partial L} - \hat{E}_{-}\right] R^{+} = {} & \chi^{+}(u^{+}, y) - æ^{+}(u^{+}, y)\, R^{+} - \\ & - R^{+}\, æ^{-}(u^{+}, y) + R^{+}\, \chi^{-}(u^{+}, y)\, R^{+}, \end{aligned} \tag{24}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_-\right] R^- = T^- \chi^-(u^+, y)\, T^+, \tag{25}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_-\right] T^- = -T^- \text{æ}^-(u^+, y) + T^- \chi^-(u^+, y)\, R^+. \tag{26}$$

The first quartet of equations is a consequence of variations of the left boundary of medium, and the second quartet is that of the right boundary. The corresponding operators of radiation "response" of medium can be written [5]

$$\hat{E}_+ = \alpha^+\left(x, u_L^-\right)\frac{\partial}{\partial x}, \quad \hat{E}_- = \alpha^-(u^+, y)\,\frac{\partial}{\partial y}, \tag{27}$$

where $\alpha^{\pm}$ are the well-known integral of collisions of the problem. Without going into details, we note that the initial conditions in the corresponding Cauchy problem, in terms of the parameter of layer thickness, are $R^{\pm}|_{L=0} = 0$, $T^{\pm}|_{L=0} = 1$, and in terms of the energy variables (x, y) – more particular solutions of PRT of single quantum, when the medium is excited by radiation incident only on one boundary (for details, see Example in Sect. 6).

5 Ambartsumian's functional equations for linear images

A third method of solution of PRT corresponds to the case when simultaneously vary both boundaries of the layer (when the elementary layer of infinitesimal thickness is added to one boundary, and it is subtracted from the other boundary). At this, a geometry of the problem is not changed, i.e. the layer thickness remained constant, so the derivatives of the spatial variables naturally should be excluded. By pairwise exclusion of derivatives over thickness from Eqs. (19)–(26), we obtain four functional equations of ACI for the linear images:

$$\begin{aligned}\hat{A}T^+ &= T^+\,\text{æ}^+\left(x, u^-\right) - \text{æ}^+(u^+, y)\, T^+ \\ &\quad + R^+\,\chi^-(u^+, y)\, T^+ - T^+\,\chi^+\left(x, u^-\right)\, R^-,\end{aligned} \tag{28}$$

$$\begin{aligned}\hat{A}R^+ &= \chi^+(u^+, y) - \text{æ}^+(u^+, y)\, R^+ - R^+\,\text{æ}^-(u^+, y) \\ &\quad + R^+\,\chi^-(u^+, y)\, R^+ - T^+\,\chi^+\left(x, u^-\right)\, T^-,\end{aligned} \tag{29}$$

$$\begin{aligned}\hat{A}R^- &= -\chi^-\left(x, u^-\right) + R^-\,\text{æ}^+\left(x, u^-\right) + \text{æ}^-\left(x, u^-\right)\, R^- - \\ &\quad - R^-\,\chi^+\left(x, u^-\right)\, R^- + T^-\,\chi^-(u^+, y)\, T^+,\end{aligned} \tag{30}$$

$$\begin{aligned}\hat{A}T^- &= \text{æ}^-\left(x, u^-\right)\, T^- - T^-\,\text{æ}^-(u^+, y) \\ &\quad + T^-\,\chi^-(u^+, y)\, R^+ - R^-\,\chi^+\left(x, u^-\right)\, T^-.\end{aligned} \tag{31}$$

The corresponding operator of radiation "response" of medium, when simultaneously vary both boundaries, i.e. the ACI operator, is given by $\hat{A} = \hat{E}_+ - \hat{E}_-$:

$$\hat{A} \equiv \alpha^+\left(x, u_L^-\right)\frac{\partial}{\partial x} - \alpha^-\left(u_L^+, y\right)\frac{\partial}{\partial y}. \tag{32}$$

It is noteworthy that the equations of linear images (20)–(31) favorably differed from the corresponding equations of previously known [5], $u_L^{\pm}(x,y)$, in the followings: (i) they retain a constructive explicit structure distinctive only for the equations of linear case, (ii) the characteristics of the elementary act of scattering (dependent on level of excitation of medium) are clearly separated from the structural forms, which are caused by the multiple scattering. The characteristics of the elementary act – $æ^{\pm}(x,y)$ and $\chi^{\pm}(x,y)$ at the transition to the linear case are converted into constant, when explicit structural forms, those just caused by the multiple scattering, are naturally retained. A transition to the functional equations of ACI (i.e. turn from the second method to the third, for determining the linear images) provides additional simplification. The layer thickness here are figured as fixed parameter for the whole calculation, whereas in the same problem with a given value of the layer thickness, the use of invariant imbedding necessarily implies an additional calculation of the entire family of PRT, starting from the value of zero thickness and continuing until reaching its final value, intended beforehand.

6 Particular example

Let us investigate next the simple instructive model of isotropic medium, with the conservative and isotropic scattering. Here we have the simplifications $R^{\pm} \equiv R$, $T^{\pm} \equiv T$, $R + T = 1$. The ACI equations, for determining the linear image T of function $u \equiv u(x,y)$, can be put in the simple symmetrical form

$$\left[k\,(x+v)\,\frac{\partial}{\partial x} + k\,(y+u)\,\frac{\partial}{\partial y}\right] T = -T\,M(x,y), \tag{33}$$

where

$$\begin{aligned} M(x,y) =& M(y,x) \equiv \frac{k\,(x+v) - k\,(y+u)}{x-y}, \\ k\,(\xi) \equiv& n\,\frac{h\nu}{2}\,\frac{A_{21}\,B_{12}}{A_{21} + \frac{\xi}{2}\,(B_{12}+B_{21})}, \end{aligned} \tag{34}$$

$$u = (x-y)\,T + y, \qquad v = -\,(x-y)\,T + x. \tag{35}$$

The initial conditions for Eq. (33) will be $T\,(x,0) = \sigma\,(x)$ or $T\,(0,y) = \sigma\,(y)$, where the unknown function $\sigma\,(z)$ describes the passage of a single quantum through medium, when it is excited by radiation of intensity z incident only on one boundary, and determined from its equation of invariant imbedding

$$\left[\frac{\partial}{\partial L} + x\,\sigma\,\frac{k\,(2\,x - x\,\sigma)}{2}\,\frac{\partial}{\partial x}\right]\sigma = -\sigma^2\,\frac{k\,(2\,x - x\,\sigma)}{2}, \tag{36}$$

$$\sigma|_{L=0} = 1, \quad \text{or} \quad \sigma|_{x=0} = q, \tag{37}$$

where q is the transmittance of layer of geometrical thickness L, in linear problem of isotropic medium at conservative isotropic scattering. It is explicitly given by

$$q = \frac{1}{1 + \frac{1}{2}\,k_0\,L}. \tag{38}$$

Thus, in this particular example, the following sequence of solutions of the problem we have in short: first solved a linear problem by means of (38), then this solution is used to define a linear image of a particular nonlinear PRT of unilateral illumination of medium by considering the auxiliary Cauchy problem (36)–(37) (by means of the equation of invariant imbedding), and afterward then the quasi-linear system of ACI (33)–(35) was considered. Hence, the desired solution of the nonlinear PRT, in term of its linear image of transmission of a single quantum, is given in an explicit form by (35).

7 Conclusion

In conclusion I want to express my deep appreciation to organizers of the conference in honor of bright memory and the 100th anniversary of academician V.V. Sobolev. For my great fascination by the theory of radiative transfer, I fully obliged to the two outstanding achievements of the field: the first is the "principle of invariance" of my teacher V.A. Ambartsumian, and the second is the "probabilistic interpretation" of V.V. Sobolev. Their incorporation provides the researchers by a powerful tools and methods of effective analysis of transfer problems, and a clear vision of their future opportunities. I am sure that the representatives of many more generations of astrophysicists, like me, would be fascinated by this area of knowledge.

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On Some Applications of General Invariance Relations Reduction Method to Solution of Radiation Transfer Problems

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Foundations of the general invariance relations reduction method are presented outline. A number of solutions of problems of the radiation transfer theory, obtained by the help of this method, is described briefly.

1 Background of the general invariance relations reduction method (GIRRM)

Properties of symmetry and invariance are widely used practically in all fields of people activity [1]. Very often these properties and principles make sense of statements on invariance of some objects, systems, equations, constructions, solutions and so on with respect to sets of actions and operations that form group. However not all properties and principles of invariance can be formulated in the framework of group-theoretical approach [1, 2]. It is necessary to point out a number of the fundamental works in which the concept of the immutability (invariance) solutions of one-dimensional (in the space variables) problems of optics and radiation transfer theory (RTT) under the simplest of the above-mentioned actions and operations are used. These publications include works written by Stokes [3], Ambartsumian (see Refs. in [4]), Chandrasekhar [5], Bellman and Kalaba [6], and Sobolev (see Refs. in [7]). The first principles of invariance in the RTT were formulated by Ambartsumian [4] and Chandrasekhar [5]. Then in 1956 Bellman and Kalaba formulated in a sufficiently abstract way the classical principle of invariant imbedding (PII). More wide interpretation, generalization and application of classical principles of invariance of the RTT were given in a number of works (see Refs. in [2, 8]). The general invariance relations reduction method (GIRRM) was proposed by Rogovtsov [1, 2, 9, 10].

The most important basic GIRRM statements and constructions are the general invariance principle (GIP) and the general invariance relations (GIRs). More narrow formulation of the GIP (in framework of the RTT) was given by Rogovtsov [2, 10]. Most universal formulation of this principle was given in the monograph [1]. By the GIRs we understand consequences of invariance (partial

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invariance) of solutions of problems of the RTT and other mathematical physics problems in respect to above-mentioned actions and operations. The GIRs connect solutions of different or the same type RTT or MP problems. It should be noted that the GIRRM are an heuristic, general and effective method

2 A short list of the results obtained by using the GIRRM

2.1 About rigorous derivation of asymptotic formulas for the plane albedo and spherical albedo for the case of nearly conservative scattering

Using rigorous integral relations and some a priori assumptions Sobolev [7] obtained the following three-term asymptotic formulas:

$$A_{pl}(\mu_1;\omega_0) \sim 1 - 4\sqrt{\frac{1-\omega_0}{3-x_1}}u(\mu_1;1) + b_{Sob}(\mu_1)(1-\omega_0), \quad \omega_0 \to 1, \tag{1}$$

$$A_{sp}(\omega_0) \sim 1 - 4\sqrt{\frac{1-\omega_0}{3-x_1}} + D(3-x_1)^{-1}(1-\omega_0), \quad \omega_0 \to 1. \tag{2}$$

Here $A_{pl}\,(\mu_1;\,\omega_0) = 2\int_0^1 \rho^0_{[0,+\infty)}\,(\mu,\mu_1;\,\omega_0)\,\mu\,d\mu$ is the plane albedo and $\rho^0_{[0,+\infty)}\,(\mu,\mu_1;\,\omega_0)$ is the azimuthally averaged reflection function for a semi-infinite plane-parallel medium (ω_0 is a single scattering albedo), $A_{sp}(\omega_0) = 2\int_0^1 \mu_1 A_{pl}(\mu_1;\omega_0)d\mu_1$ is the spherical albedo, $u(\mu;\omega_0)$ is the function that defines the angular dependence of Milne's problem solution [7, 8], $b_{Sob}(\mu_1) = 15(5-x_2)^{-1}\big(\mu_1^2 - 2\int_0^1 \rho^0_{[0,+\infty)}(\mu,\mu_1;1)\mu^3 d\mu\big) + D(3-x_1)^{-1}u(\mu_1;1)$, where $D = 24\int_0^1 u(\mu;1)\mu^2 d\mu$, $\{x_s\}_{s\in N_0}$ is a sequence of expansion coefficients of the phase function $p(\mu)$ in Fourier series in the system of Legendre polynomials $\{P_s(\mu)\}_{s\in N_0}$ ($p(\mu) = \sum_{s=0}^{+\infty} x_s P_s(\mu); N_0 = \{0,1,2,\dots\}$). Without any a priori assumptions, using the GIRRM, we were strictly obtained Eq. (2) and such asymptotic formula:

$$A_{pl}(\mu_1;\omega_0) \sim 1 - 4\sqrt{\frac{1-\omega_0}{3-x_1}}u(\mu_1;1) + b_{R,B}(\mu_1;\omega_0)(1-w_0), \quad \omega_0 \to 1. \tag{3}$$

The value of $b_{R,B}(\mu_1;\omega_0)$ in Eq. (3) is equal to

$$\begin{aligned} b_{R,B}(\mu_1;\omega_0) = {} & D(3-x_1)^{-1}u(\mu_1;1) \\ & - 15(5-x_2)^{-1}\left[\mu_1^2 - 2\int_0^1 \mu^3\rho^0_{[0,+\infty)}(\mu,\mu_1;1)d\mu\right] \\ & - 2\int_{-1}^1 d\mu \int_{+0}^{+\infty} \left[\tilde{G}^*_{\infty;0}(\tau,\mu;0,\mu_1;\omega_0) - J(\tau,\mu,\mu_1;\omega_0)\right] d\tau, \end{aligned} \tag{4}$$

$$J(\tau,\mu,\mu_1;\omega_0) = 2\int_0^1 \mu'\tilde{G}^*_{\infty;0}(\tau,\mu,0,-\mu';\omega_0)\rho^0_{[0,+\infty)}(\mu',\mu_1;1)d\mu'.$$

Table 1: Numerical values of the coefficients $b_{Sob}(\mu_1)$ and $b_{R,B;0}(\mu_1)$

μ_1	$g = 0.65$	$g = 0.75$	$g = 0.85$	$g = 0.97$	$g = 0.991$
0.0381347	1.613	2.339	3.901	18.37	60.14
0.238853	3.212	4.652	7.974	41.00	137.3
0.333212	4.080	5.863	9.993	51.15	171.1
0.434867	5.114	7.289	12.34	62.78	209.8
0.539374	6.285	8.894	14.97	75.66	252.6
0.642166	7.545	10.61	17.77	89.32	298.0
0.738751	8.829	12.60	20.60	103.1	343.7
0.824908	10.05	14.02	23.30	116.1	387.0
0.896871	11.14	15.49	26.68	127.6	425.0
0.951494	12.00	16.65	27.54	136.7	455.0
1.0	12.78	17.72	29.26	144.9	482.4

In the formula (4) the function $\tilde{G}^{*}_{\infty;0}(\tau,\mu;\tau',\mu';\omega_0)$ has the meaning of not the main part of the contribution to the integrated over azimuth "volume" Green function [2, 11] of the dimensionless scalar radiative transfer equation (SRTE) for the case of an infinite plane-parallel medium. This part is generated by the subset of the spectrum of the reduced characteristic equation of the SRTT (it corresponds to zero azimuthal harmonic of phase function). The above-mentioned subset does not contain only the minimum in modulus eigenvalues. The asymptotic formulas (1) and (3) differ in shape (their third members have different forms). Nonetheless theoretical analysis and the series of numerical experiments showed that there are no differences (within the limits of calculation errors for sufficiently small values of q ($q = 1-\omega_0$) between the coefficients $b_{Sob}(\mu_1)$ and $b_{R,B}(\mu_1;\omega_0)$.

Remark 1. *The coefficient $b_{R,B}(\mu_1;\omega_0)$ can be represented in form of the following series:*

$$b_{R,B}(\mu_1;\omega_0) = \sum_{l=0}^{+\infty} b_{R,B;l}(\mu_1)(1-\omega_0)^l. \tag{5}$$

This series is convergent point-wise and uniformly on [-1,1] for sufficiently small values of q.

Remark 2. *Using the GIRRM the effective analytical and numerical algorithms for finding all the quantities in the asymptotic formulas* (2) *and* (3) *for any phase functions are developed.*

In Table 1 a number of numerical values of the coefficients $b_{Sob}(\mu_1)$ and $b_{R,B;0}(\mu_1)$ are given for the case of Henyey-Greenstein's phase function $\chi(\mu;g)$ [7, 8]. From the above-said and Table 1, it follows that the assumptions [7], which V.V. Sobolev used in the derivation of the formulas (1) and (2), are correct for situations considered.

2.2 The correct methods of derivation of multi-term asymptotics for the case of plane-parallel media

The GIRRM allows to derive asymptotic formulas for azimuthally averaged reflection and transmission coefficients [7] for the case of plane-parallel optically thick media without using a priori assumptions about their structures. It should be noted that it is necessary to take into account the implicit contribution of the entire spectrum of the characteristic equation (CE) of the SRTE in the above-mentioned coefficients in the process of rigorous derivation of these asymptotics. In particular, all of the elements (they belong to the spectrum of the CE), which do not coincide with minimal in modulus elements of the same spectrum, contribute some terms of the order of $(1-\omega_0)$ (if $\omega_0 \to 1$) to asymptotics of these coefficients. Using some constructions of the GIRRM, the principle of reciprocity [11] and the analytical representations (see [12, 13]) for the "volume" Green function of the SRTE for a infinite plane-parallel medium, we have proved the faithfulness of Sobolev's a priori assumptions and asymptotic formulas [7] for the above-mentioned coefficients for the cases of semi-infinite media and layers of a large optical thickness τ_0. In [14] multi-term formulas for the reflection and transmission coefficients when $\tau_0 \to \infty$ were first obtained in implicit form. These asymptotics were found by using Case's method. Then the methods of finding of multi-terms asymptotics of various radiative characteristics were proposed in [15, 16, 17]. The most effective algorithm for deriving of such asymptotics was described in [15]. This algorithm was based on the constructive ideas of the GIRRM. To illustrate capabilities of this algorithm we write down only some relations from [15]. Consider a macroscopically homogeneous and local isotropic plane-parallel turbid layer of an optical thickness τ_0. Then using standard constructions of the GIRRM [1, 2] the following GIRs:

$$\rho^0(|\mu|,\xi;\omega_0,\tau_0) = g_1^0(|\mu|,\xi;\omega_0,\tau_0) + \int_0^1 K(|\mu|,\mu'';\omega_0,\tau_0)\rho^0(\mu'',\xi;\omega_0,\tau_0)d\mu'', \quad (6)$$

$$\sigma^0(|\mu|,\xi;\omega_0,\tau_0) = g_2^0(|\mu|,\xi;\omega_0,\tau_0) + \int_0^1 K(|\mu|,\mu'';\omega_0,\tau_0)\sigma^0(\mu'',\xi;\omega_0,\tau_0)d\mu'', \quad (7)$$

$$(|\mu|,\xi \in [0,1],\ \omega_0 \in [0,1],\ \tau_0 \in (0,+\infty))$$

were obtained in [15]. In GIRs (6) and (7) the functions $\rho^0(|\mu|,\xi;\omega_0,\tau_0)$ and $\sigma^0(|\mu|,\xi;\omega_0,\tau_0)$ are the azimuthally averaged reflection and transmission coefficients [7, 8] of a layer correspondingly. The function $K(|\mu|,\mu'';\omega_0,\tau_0)$ is defined by the relation.

$$K(|\mu|,\mu'';\omega_0,\tau_0) = \mu'' \int_0^1 \mu' \tilde{G}_{[0,+\infty)}(0,-|\mu|;\tau_0,\mu';\omega_0)\tilde{G}_{[0,+\infty)}(0,-\mu';\tau_0,\mu'';\omega_0)d\mu'. \quad (8)$$

Here function $\tilde{G}_{[0,+\infty)}(\tau,\mu;\tau',\mu';\omega_0)$ is the "volume" Green function of the dimensionless SRTE for the case of a semi-infinite plane-parallel medium which comprises the "sources" $\delta(\tau - \tau')\delta(\mu - \mu')$ $(\tau' > 0)$. In turn the functions $g_1^0(|\mu|, \xi; \omega_0, \tau_0)$, $g_2^0(|\mu|, \xi; \omega_0, \tau_0)$ can be expressed in terms of values of this "volume" Green function (see [15]).

For example, using the principle of reciprocity [11], the representations for the Green function $\tilde{G}_{[0,+\infty)}(\tau, \mu; \tau', \mu'; \omega_0)$ [12, 13], K-integral of the SRTE [7] and GIRs (6), (7) the following asymptotics:

$$\begin{aligned}\sigma^0(|\mu|, \xi; 1, \tau_0) = {} & Q(|\mu|, \xi; \tau_0) + (2u(|\mu|; 1) + h_2(|\mu|; \tau_0))\gamma_1(\tau_0, x_1) \\ & \times \left\{ \int_0^1 \mu'^2 \rho^0_{[0,+\infty)}(\mu', \xi; 1) d\mu' + h_1(\xi; \tau_0) + \gamma_2(\tau_0, \xi, x_1) \right\} \\ & + O(\tau_0^{-1} \exp(-2k_2\tau_0)), \quad \tau_0 \to +\infty, \\ \gamma_1(\tau_0, x_1) = {} & \left[(1 - \frac{x_1}{3})\tau_0 + 4 \int_0^1 \mu'^2 u(\mu'; 1) d\mu' + h(\tau_0) \right]^{-1}, \\ \gamma_2(\tau_0, \xi, x_1) = {} & 2^{-1}\xi\Big(1 - \exp\Big(-\frac{\tau_0}{\xi}\Big)\Big) - 2^{-1}\Big(1 - \frac{x_1}{3}\Big)\tau_0 \exp\Big(-\frac{\tau_0}{\xi}\Big),\end{aligned} \tag{9}$$

$$\begin{aligned}\int_0^1 \mu\sigma^0(\mu, \xi; 1, \tau_0) d\mu = \gamma_1(\tau_0, x_1) & \left\{ \int_0^1 \mu'^2 \rho^0_{[0,+\infty)}(\mu', \xi; 1) d\mu' \right. \\ & \left. + h_1(\xi; \tau_0) + \gamma_2(\tau_0, \xi, x_1) \right\} \\ & + O(\tau_0^{-2} \exp(-2k_2\tau_0)), \quad \tau_0 \to +\infty,\end{aligned} \tag{10}$$

were obtained in [15]. The functions $Q(|\mu|, \xi; \tau_0)$, $h(\tau_0)$, $h_1(\xi; \tau_0)$, $h_2(|\mu|; \tau_0)$ are expressed explicitly in terms of the functions $u(|\mu|; 1)$, $\rho^0_{[0,+\infty)}(|\mu|, \xi; 1)$, $\tilde{G}^*_{\infty;0}(\tau, \mu; \tau', \mu'; \omega_0)$. In addition there are asymptotics $h(|\mu|; \tau_0) = O(\exp(-k_2\tau_0))$, $h_1(\xi; \tau_0) = O(\exp(-k_2\tau_0))$, $h_2(|\mu|; \tau_0) = O(\exp(-k_2\tau_0))$, $\tau_0 \to +\infty$. In Eqs. (9), (10) under symbol k_2 it should be understood the second non-negative root of the reduced characteristic equation of the SRTE (if it exists). If a root does not exist under the symbol k_2 it is necessary to understand the positive number $(1 - \varepsilon)$, where ε is a small enough positive number. Eq. (9) is a generalization of asymptotics for $\sigma^0(|\mu|, \xi; 1, \tau_0)$ obtained by Sobolev [7].

2.3 Constructive theory of scalar characteristic equations of the radiative transfer theory

The constructive theory of scalar characteristic equations of the RTT was suggested in [12, 13, 18]. The construction of solutions of these equations in analytic form can be reduced to finding solutions of infinite tridiagonal systems of linear algebraic equations. Effective analytical and numerical algorithms for finding discrete spectra, eigenfunctions and normalizing constants for reduced scalar characteristic equations of the SRTE was described in above-mentioned works. New two-term recursion formulas and analytic representations for solutions of infinite tridiagonal systems of linear algebraic equations were suggested in [13]. In addition, Rogovtsov obtained a general analytic expression for the "volume"

Green function of a two-dimensional (with respect to the angular variables) integro-differential equation of the radiative transfer for the case in which the phase function satisfies the Hölder condition on [-1, 1].

2.4 Effective algorithms for finding the reflection function, plane and spherical albedo for any phase function

Properties of invariance are used in the RTT in developing the effective algorithms for finding the reflection function, plane and spherical albedo. Point out two algorithms, in which these properties are used in an explicit form. The first algorithm is based on the use of Ambartsumian's non-linear integral equations for the reflection function and its azimuthal harmonics. The second algorithm was developed through the use of Fredholm special integral equations. The nonlinear integral above-mentioned equations were obtained by Ambartsumian by using the principle of invariance which he formulated in 1943 (see Refs. in [4]). Special Fredholm equations were found through the use of rigorous mathematical considerations or some properties of invariance in a number of papers (see, for example, [2, 12, 19, 20, 21] and references therein). The first algorithm was used, in particular, in [22]. The second algorithm is actually used in [2, 12]. It should be noted that the correct application of both algorithms requires the use of additional information about solutions of other problems of the RTT. For example, the quantities describing the deep regime of the radiation intensity in a semi-infinite medium and the Sobolev–van de Hulst relation [7, 8, 22] was used in the first algorithm [22] as an additional information in the construction of a sustainable iterative algorithm for solving nonlinear scalar Ambartsumian's integral equations. Previously it is necessary to find "volume" Green function of the SRTE for the case of an infinite plane-parallel turbid medium if special Fredholm integral equations are taken as the initial equations when finding of the reflection function. Before developing effective analytical and numerical algorithms for finding the above-mentioned Green function for cases of arbitrary phase functions it was practically impossible to use this kind of equations. Such algorithms were constructed and effectively used in [2, 12, 13]. These algorithms can be applied for the cases of sharply anisotropic phase functions. To illustrate the effectiveness of the algorithms developed in [2, 12, 13] we give below Table 2 for the quantities $A_{pl}(\mu_1;\omega_0)$, $A_{sp}(\omega_0)$ for the case of the phase function $\chi(\mu;g)$.

2.5 Exact expressions, asymptotic formulas, inequalities and asymptotic inequalities for the average characteristics of radiative fields in turbid media of different configurations

Different GIRs can be used for finding the integral invariants of the stationary and non-stationary SRTE. Moreover a number of average characteristics of radiative fields in turbid media of different configurations can be found using the GIRs. In the most simple form such results were obtained by Rogovtsov (see [1, 2, 13] and

Table 2: Values for plane and spherical albedo $A_{pl}(\mu_1;\omega_0)$, $A_{sp}(\omega_0)$ ($g = 0.993$)

μ_1	$\omega_0 = 0.99$	0.993	0.997	0.999	0.9999	0.99999
0.844195×10^{-2}	0.6995	0.7326	0.7996	0.8668	0.9504	0.9834
0.381347×10^{-1}	0.5356	0.5833	0.6836	0.7879	0.9206	0.9734
0.880185×10^{-1}	0.4097	0.4643	0.5860	0.7191	0.8939	0.9643
0.155914	0.3135	0.3670	0.5027	0.6575	0.8691	0.9558
0.238853	0.2407	0.2948	0.4314	0.6016	0.8453	0.9476
0.434867	0.1456	0.1906	0.3202	0.5057	0.8010	0.9318
0.642166	0.09391	0.1294	0.2442	0.4312	0.7622	0.9174
0.738751	0.07811	0.1097	0.2170	0.4020	0.7456	0.9111
0.896871	0.05896	0.08970	0.1803	0.3598	0.7200	0.9011
1.0	0.04964	0.07264	0.1605	0.3353	0.7040	0.8948
$A_{sp}(\omega_0)$	0.1079	0.1431	0.2542	0.4351	0.7612	0.9167

Refs. in therein) for the cases turbid media having forms of layer, sphere, infinite circular cylinder and regular polyhedral. In these works the average duration of the luminescence and radiative fluxes were required quantities. In turn the asymptotic inequalities for the mean intensity of the radiation, the average number of scattering of a photon, the average density of radiation, radiative fluxes and spherical albedo were found by Rogovtsov, Karpuk and Samson (corresponding Refs. are given in [2, 13]). These authors considered the process of radiative transfer in turbid media that have the forms of layer, sphere, finite and infinite circular cylinders, spheroids, spherical shell and non-concavity body bounded by a smooth boundary. In some of the above-mentioned publications the presence of underlying surfaces was allowed.

2.6 On the asymptotic expressions for the Green functions of the SRTE when turbid medium contains mono-directional point or line sources

2.6.1. Let turbid "medium" $\tilde{V}$ be a macroscopically homogeneous or two-layer non-conservative semi-infinite "medium" $\tilde{V}_{[0,\infty)}$, which is irradiated by an infinitely narrow mono-directional beam of radiation or contains near its border $\tilde{S}_{[0,\infty)}$ a point mono-directional source (see Figs. 1a, 1b).

Then the intensity of the radiation (or Green functions) at an optical depth τ_0 (when $\tau_0 \to +\infty$) at any observation point P can be represented in a simple analytical form (see [23]). In addition, the principle terms of asymptotic formulas are expressed in terms of elementary functions and solutions of special BVPs for the case of a plane-parallel anisotropic absorbing semi-infinite turbid "medium".

Remark 3. *Under the above-mentioned assumptions the forms of relative intensities for deep regime behaviors tend asymptotically to each other when a semi-infinite turbid medium is irradiated by an infinitely wide mono-directional beam of radiation or infinitely narrow mono-directional beams of radiation.*

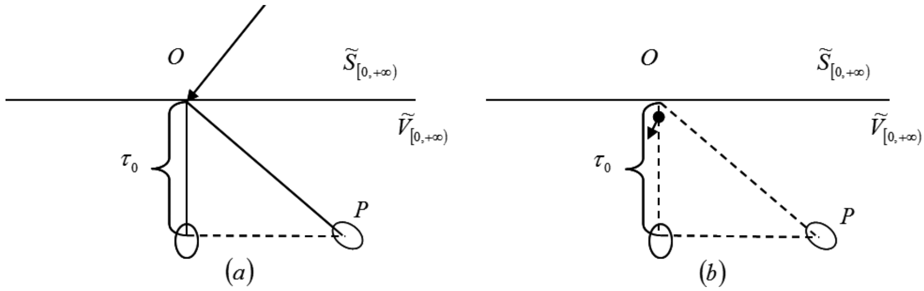

Figure 1: Geometries of problems for the cases of external and internal sources of radiation.

Remark 4. *Let the observation point P be at a large optical depth τ_0 and the shortest optical distance from point P to a perpendicular to $\tilde{S}_{[0,\infty)}$ (it passes through the point of incidence of external radiation) is equal τ_1. Then for the case of a macroscopically homogeneous turbid medium the principle term of the asymptotics of "volume" Green function of the dimensionless SRTE will be in form [24]*

$$\begin{aligned}&\tilde{G}_{[0,+\infty)}\big(\vec{\tau},\vec{\Omega};\vec{0},\vec{\Omega}_1;\omega_0\big) \sim \frac{k_1\exp(-k_1\tau_0)}{2\pi^2\tau_0}\, i(\mu;\omega_0)\, u(\mu_1;\omega_0),\\ &\omega_0\in(0,1),\quad \tau_0\to+\infty,\quad (\tau_1/\sqrt{\tau_0})\to 0\end{aligned} \tag{11}$$

Here functions $i(\mu;\omega_0)$ and $u(\mu_1;\omega_0)$ are the classic functions of the SRTT [7, 8]; k is the smallest positive element of the discrete spectrum of characteristic equation of the SRTT [7, 8, 13].

2.6.2. Let $\tilde{V}_{[0,\tau_0]}$ be a macroscopically uniform non-conservative scattering "layer" of an optical thickness τ_0 which is irradiated by a mono-directional infinitely narrow beam of radiation (see Fig. 2). Then with the help of the GIRRM the principle term of asymptotics of the "surface" Green function [11] of the dimensionless the SRTE for any position of observation point P, which is on the second boundary $\tilde{S}_2$ of the layer $\tilde{V}_{[0,\tau_0]}$, can be found. Here τ_0 tends to $+\infty$.

Remark 5. *Let the shortest optical distance from an observation point P to the perpendicular to $\tilde{S}_1$ which passes through the point of incidence of external*

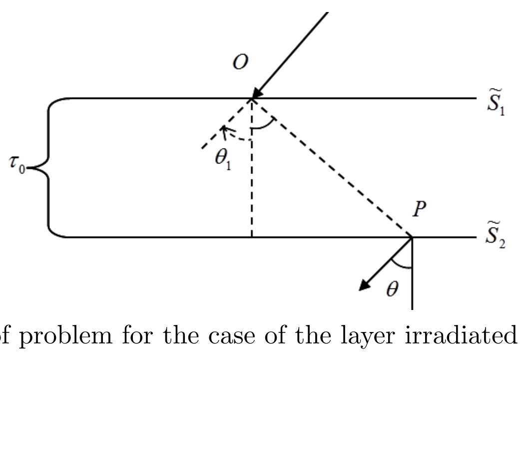

Figure 2: Geometry of problem for the case of the layer irradiated by the external beam of radiation.

radiation be equal to τ_1. Then the principle term of the asymptotics of the "surface" Green function $\tilde{G}_{\tilde{S}}(\tau, \vec{\Omega}; \vec{0}, \vec{\Omega}_1; \omega_0; \tilde{V}_{[0,\tau_0]})$ of the SRTE can be presented in the form [24]

$$\begin{aligned}
&\tilde{G}_{\tilde{S}}(\vec{\tau}, \vec{\Omega}; \vec{0}, \vec{\Omega}_1; \omega_0; \tilde{V}_{[0,\tau_0]}) \sim \mu_1 \frac{M k_1 \exp(-k_1 \tau_0)}{2\pi^2 \tau_0} u(\mu; \omega_0)\, u(\mu_1; \omega_0),\\
&\tau_0 \to +\infty, \quad \frac{\tau_1}{\sqrt{\tau_0}} \to 0, \quad \omega_0 \in (0,1), \\
&M = 2\int_{-1}^{1} \mu\, i^2(\mu; \omega_0) d\mu, \quad \mu = \cos\theta, \quad \mu_1 = \cos\theta_1.
\end{aligned} \tag{12}$$

2.6.3. Let $\tilde{V}$ be a non-conservative scattering "medium" having the shape of a sphere, the center of which is the point mono-directional "source" $\delta(\vec{\tau})\delta(\vec{\Omega}-\vec{\Omega}_1)$. In addition, the optical radius of $\tilde{V}$ is equal to τ_0. Then the asymptotic formula (see Refs. in [1, 2])

$$\tilde{G}(\vec{\tau}, \vec{\Omega}; \vec{0}, \vec{\Omega}_1; \omega_0; \tilde{V}) \sim \frac{k_1 \exp(-k_1 \tau_0)}{2\pi^2 \tau_0} u((\vec{n}\cdot\vec{\Omega}))\, i((\vec{n}\cdot\vec{\Omega}_1)), \tag{13}$$

$$\tau_0 \to +\infty, \quad ((\vec{n}\cdot\vec{\Omega}) \geq \varepsilon > 0)$$

holds. Here $\vec{n}$ is the unit dimensionless external normal to the boundary $\tilde{S}$ of the "medium" $\tilde{V}$ in a observation point (it is specified by an optical radius-vector $\vec{\tau}$) which lies at this boundary.

2.6.4. Let $\tilde{V}$ be a non-conservative scattering "medium" which has the shape of an infinite circular cylinder and contains (on the axis of symmetry) a linear mono-directional "source" $\delta(\vec{x})\delta(\vec{y})\delta(\vec{\Omega}-\vec{\Omega}_1)$ (see Fig. 3). Then the asymptotic formula (see Refs. in [1, 2])

$$\int_{-\infty}^{+\infty} \tilde{G}(\vec{\tau}_p, \vec{\Omega}; \tilde{z}\vec{e}_3, \vec{\Omega}_1; \omega_0; \tilde{V}) d\tilde{z} \sim \frac{1}{\pi}\sqrt{\frac{k_1}{2\pi\tau_0}} \exp(-k_1\tau_0)\, u((\vec{n}\cdot\vec{\Omega})\, i(\vec{n}\cdot\vec{\Omega}_1)), \tag{14}$$

$$\tau_0 \to +\infty \quad ((\vec{n}\cdot\vec{\Omega}) \geq \varepsilon > 0), \quad \omega_0 \in (0,1)$$

holds. Here $\vec{e}_3$ is the unit dimensionless vector which defines the direction of $\tilde{Z}$-axis of a dimensionless Cartesian right rectangular coordinate system $O\tilde{X}\tilde{Y}\tilde{Z}$ (the axis $\tilde{Z}$ coincides with symmetry axis of this cylinder), $\vec{\tau}_p$ specifies an observation point P, which is on the boundary of the cylinder.

2.6.5. Let V be a non-conservative scattering medium, which has a disk shape (see Fig. 4). We will assume that the local optical characteristics of V can depend only on the depth z in a Cartesian right rectangular coordinate system $OXYZ$. Assume that a plane OXY is parallel to the plane parts of the boundary of the disk V and the point O is situated on the axis of symmetry of the disk (this point should be situated inside the disk). Let the disk V contain a point isotropic "source" $\delta(\vec{r})$, which is located at the point O.

Using the GIRRM an asymptotic formula for the "volume" Green function was obtained when the observation point P is situated on the lateral boundary

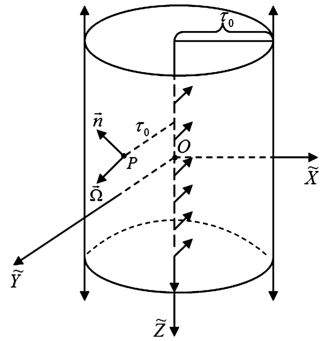

Figure 3: Geometry of the problem for the case of the infinite circular cylinder

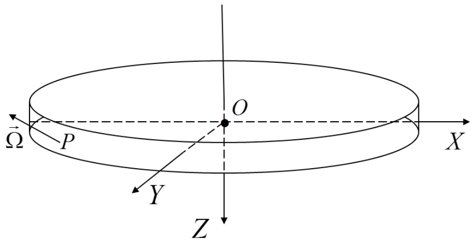

Figure 4: Geometry of the problem for the case of the disk

of the disc and the radius R of the disk tends to $+\infty$. This asymptotics has the following form (see Refs. in [1, 2]):

$$\tilde{G}(\vec{r}_p, \vec{\Omega}; \vec{0}; V) \sim \frac{c_1}{\sqrt{R}} \exp(-k^* R)\, B(z; \vec{\Omega}), \quad R \sup_{z\in[a,b]}\{\alpha(z)\} \to +\infty. \tag{15}$$

Here k^* is the smallest positive root of the non-classical characteristic equation of the SRTT, the constant c_1 is expressed through the first eigenvalue and the corresponding eigenfunction of this equation, $\alpha(z)$ is an attenuation coefficient, the function $B(z; \vec{\Omega})$ is expressed through solutions of one-dimensional and two-dimensional (in space variable) BVPs (the initial BVP is three-dimensional).

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On the Complex Radiative Transfer in an Optically Finite Homogeneous Atmosphere

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In this paper we consider the classical problem in radiative transfer – the planetary problem – in an isotropically scattering homogeneous optically finite medium where the albedo of single scattering may be defined anywhere in the complex plane.

To solve this problem we use the method of approximating the kernel in the integral equation for the Sobolev resolvent function. This approach allows to define easily determinable auxiliary functions which help us to express almost all the relevant functions of transfer for this problem.

1 Statement of the problem

Usually the albedo of single scattering λ or c is assumed real in radiative transfer. But when we consider the Laplace transformed time-dependent transport equation λ may turn complex. We met another such problem when we tried to determine the photon path-length distribution function in an optically semi-infinite atmosphere. It appeared that the non-linear integral equation for the complex H-function is valid even in the complex plane [1]. This interesting fact directed the author to a deeper treatment of the problem and to try to find for such a case the radiation field in general.

In order to solve this problem we used the kernel approximation method first proposed by Krook [2] and later developed by Rybicki [3] and Vainikko et al. [4]. Rybicki approximated the kernel in the integral equation – the exponential integral – for the Sobolev resolvent function Φ by a Gauss-Legendre sum while Krook and Vainikko et al. used the method of kernel approximation for the integral equation of the source function.

The substitution of the kernel by a Gauss-Legendre sum allows us to solve the obtained approximate equation for the Sobolev resolvent function Φ [5] exactly while the solution is a weighted sum of exponents. This approach allowed us to define simple auxiliary functions for determining the radiation field.

Here we have chosen to consider the planetary problem in a homogeneous isotropically scattering optically finite atmosphere where the albedo of single scattering is complex

$$\lambda = \lambda_1 + i\lambda_2. \tag{1}$$

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We assume that in this case the source function B for a large range of radiative transfer problems in given type of atmospheres can still be described by the well-known Fredholm integral equation

$$B(\tau, \kappa, T) = \frac{1}{2}\lambda \int_0^T E_1(t-\tau) B(t, \kappa, T) dt + \frac{1}{4}\lambda F \exp(-\tau/\kappa), \tag{2}$$

where τ is the optical depth, T is the optical thickness of the atmosphere, πF is the flux of the incident radiation normal to the plane of stratification, κ is the direction cosine of the angle of incidence referred to the outward normal of the atmosphere and the exponential integral is expressed in the form

$$E_n(x) = \int_0^1 \exp(-|x|/s) s^{n-2} ds. \tag{3}$$

Since the Sobolev resolvent function is the regular part of the Green function for Eq. (2) we may immediately write that the solution of Eq. (2) for the planetary case is

$$B(\tau, \kappa, T) = \frac{1}{4}\lambda F \left[\exp(-\tau/\kappa) + \int_0^\tau \Phi(t, T) \exp(-t/\kappa) dt \right], \tag{4}$$

where the Sobolev resolvent function satisfies the following Fredholm integral equation [6]

$$\Phi(\tau; T) = \frac{1}{2}\lambda \int_0^T E_1(t-\tau)\, \Phi(t; T) dt + \frac{1}{2}\lambda E_1(\tau). \tag{5}$$

Now the task is all set for the solution of Eq. (5) by approximating the kernel of this equation.

2 Solution of the equation for the approximate resolvent function

Eq. (5) is one of the most important equations in the radiative transfer since all the relevant functions of transfer can be expressed through the resolvent function.

We try to solve Eq. (5) by approximating the kernel of it by a sum of exponents

$$E_1(\tau) = \sum_{n=1}^{N} w_n \exp(-\tau/u_n) u_n^{-1}, \tag{6}$$

where w_n are the weights and u_n are the points of a Gauss quadrature rule of the order of N in the interval (0,1). After substitution of this approximation into

Eq. (5) the resulting equation can be solved exactly and the solution is

$$\Phi(\tau, T) = a_1 + b_1\tau + \sum_{i=2}^{N} \{a_i \exp(-s_i\tau) + b_i \exp[s_i(T-\tau)]\}. \tag{7}$$

If $\lambda_1 \neq 1$ then $a_1 = b_1 = 0$ outside of the summation sign and the summation begins at $i = 1$. This rule applies throughout the paper. The unknown coefficients s_i are the zeros of the equation

$$1 - \lambda \sum_{n=1}^{N} \frac{w_n}{1 - s^2 u_i^2} = 0. \tag{8}$$

The approximate characteristic equation – Eq. (8) – can simply be solved when λ is real and positive since we know beforehand in which intervals to search for the zeros. This is not the case when λ is complex or negative but if we write Eq. (8) in the polynomial form

$$\sum_{i=1}^{N} c_i s_i^{2i} = 0, \tag{9}$$

we may apply the code DZROOTS from Numerical Recipes [7].

The coefficients a_i and b_i are to be found from linear algebraic systems of equations

$$\begin{aligned} \alpha_1 + \sum_{i=2}^{N} \alpha_i \left[\frac{1}{1 - s_i u_j} + \frac{\exp(-s_i T)}{1 + s_i u_j}\right] &= u_j^{-1}, \\ \beta_1(T + 2u_j) + \sum_{i=2}^{N} \beta_i \left[\frac{1}{1 - s_i u_j} - \frac{\exp(-s_i T)}{1 + s_i u_j}\right] &= u_j^{-1}, \quad j = 1, 2, \ldots, N, \end{aligned} \tag{10}$$

while

$$\begin{aligned} &a_i = (\alpha_i + \beta_i)/2, \quad b_i = (\alpha_i - \beta_i)/2, \quad i = 2, \ldots, N; \\ &a_1 = (\alpha_1 - \beta_1 T)/2, \quad b_1 = \beta_1. \end{aligned} \tag{11}$$

This system may be solved, e.g., using algorithms ZGECO and ZGESL from LINPACK.

Thus, the solution of the approximate equation for the Sobolev resolvent function is completed.

3 The complex radiation field

Next we define the auxiliary functions x and y as the generalizations of the well-known Ambartsumian–Chandrasekhar functions X and Y

$$x(\tau,\kappa,T) = 1 + \int\limits_{\tau}^{T} \Phi(t;T) \exp\left[-(t-\tau)/\kappa\right] dt,$$

$$y(\tau,\kappa,T) = \exp\left(-\tau/\kappa\right) + \int\limits_{0}^{\tau} \Phi(t;T) \exp\left[-(\tau-t)/\kappa\right] dt. \tag{12}$$

By the use of these functions we may express the solution of Eq. (2) [8] as

$$B(\tau,\kappa,T) = \frac{1}{4}\lambda F\{X(\kappa,T)y(\tau,\kappa,T) - Y(\kappa,T)[x(T-\tau)-1]\}. \tag{13}$$

Evidently, the Ambartsumian–Chandrasekhar functions X and Y are the special cases of x and y

$$\begin{aligned} X(\kappa,T) &= x(0,\kappa,T), \\ Y(\kappa,T) &= y(T,\kappa,T). \end{aligned} \tag{14}$$

As the next step we use the well-known definitions for the intensities and find that for the upward moving radiation, i.e. for the intensities towards the smaller optical depths we have

$$\begin{aligned} I(\tau,-\mu,\kappa,T) = \frac{\lambda F}{4}\frac{\kappa}{\mu+\kappa}\{&X(\kappa,T)[x(\tau,\mu,T) + y(\tau,\kappa,T) - 1] \\ &- Y(\kappa,T)[x(T-\tau,\kappa,T) + x(T-\tau,\mu,T) - 1]\} \end{aligned} \tag{15}$$

and for the intensities towards the larger optical depths

$$\begin{aligned} I(\tau,\mu,\kappa,T) = \frac{\lambda F}{4}\frac{\kappa}{\mu-\kappa}\{&X(\kappa,T)[y(\tau,\mu,T) - y(\tau,\kappa,T)] \\ &- Y(\kappa,T)[x(T-\tau,\mu,T) - x(T-\tau,\kappa,T)]\}. \end{aligned} \tag{16}$$

The seeming discontinuity in Eq. (16) may be eliminated by the L'Hopital rule.

4 Results

We have performed calculations for different set of atmospheric parameters and we are convinced that at least for the region $-8 \le \lambda_1 \le 8$ and $-8 \le \lambda_2 \le 8$ our method works well. We checked the results by solving the Ambartsumian–Chandrasekhar differential equations [9] for X and Y functions and coincidence of the results even for the modest $N = 7$ was very good.

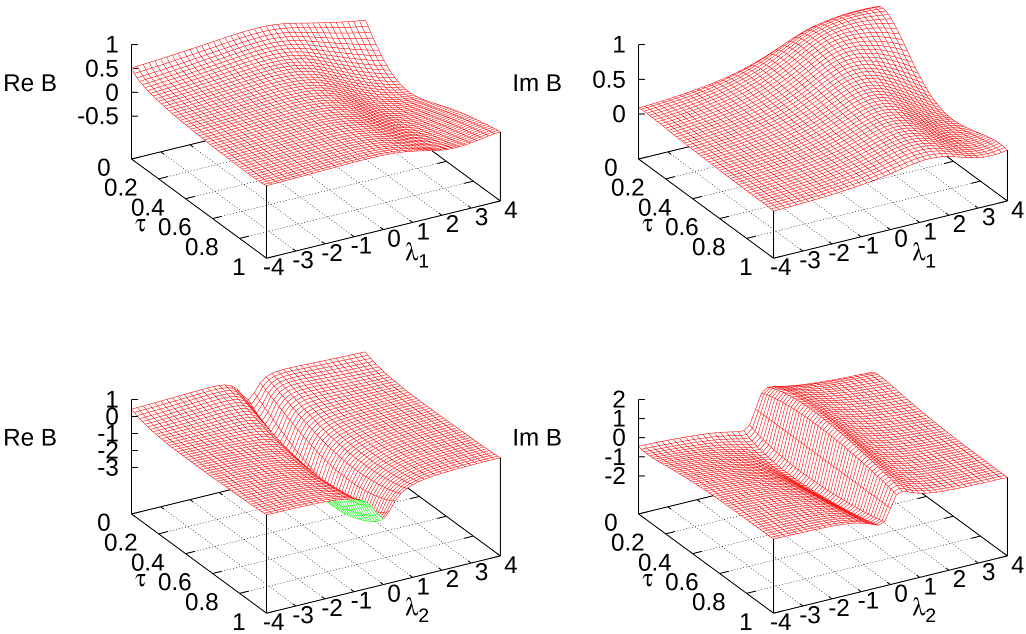

Figure 1: The real and imaginary parts of the source function ($\kappa = 1.0$, $T = 1.0$). Upper panel: $\lambda_2 = 2$, lower panel: $\lambda_1 = 2$.

In Fig. 1 we have presented some results for the source function. One may notice that for the fixed λ_2 the surfaces – both for the real and imaginary parts – of B are quite smooth while these for the fixed λ_1 have jumps at $\lambda_2 = 0$. We met a similar behavior when computing the complex H function [1].

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* The color figure is available online in the Proceedings at *http://www.astro.spbu.ru/sobolev100/*.

Polarization of Resonance Lines in the Case of Polarized Primary Sources of Radiation

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Transfer of polarized radiation in a spectral line in a non-magnetic semi-infinite plane-parallel atmosphere is considered. Complete frequency redistribution is assumed. It is supposed that primary sources of the radiation distributed in the atmosphere are partially polarized. The dependence on the optical depth of these sources is described by the product of a polynomial in the exponent. The problem is to find the radiation emergent from the atmosphere. The general theory of $\widehat{\mathbf{I}}$-matrices is applied to this problem. It turns out that the solution of the problem with any of the primary sources of this type is reduced to the solution of the so-called standard problem, and the subsequent simple numerical integration.

We consider multiple resonance scattering of radiation in a spectral line that takes place in a semi-infinite plane-parallel atmosphere without a magnetic field. Due to the symmetry, the radiation field can be described by two Stokes parameters I and Q. Therefore, the scattering is completely described by the two-component Stokes vector $\mathbf{i}(\tau, x, \mu) = (I, Q)^T$; its arguments are the usual optical depth averaged over line τ, the dimensionless frequency measured from the center of the line x, and the cosine of the zenith angle μ. Also, complete frequency redistribution is assumed. There are polarized primary sources of radiation embedded in the atmosphere; we suggest they are given by the vector function

$$\mathbf{s}_k(\tau) = \tau^k \, e^{-\tau/z_0} \, \mathbf{s}_0, \quad k = 0, 1, 2, \ldots, \tag{1}$$

where $z_0 \in (0, \infty)$ is a parameter, and $\mathbf{s}_0$ is a known constant vector.

In the works [1, 2] the theory of $\widehat{\mathbf{I}}$-matrices was developed. This theory makes possible to generalize a number of well-known results of the standard scalar theory of line formation to the problems when polarization of the radiation is taken into account. The scalar version of the problem considered here was studied in [3]. All the details regarding the theory of $\widehat{\mathbf{I}}$-matrices, for example, the relation of the matrix transfer equation to the vector one, as well as designations used here can be found in [1, 2].

By definition, the Stokes matrix $\widehat{\mathcal{I}}(\tau, z)$ is a solution of the matrix transfer equation

$$z \, \frac{\partial \widehat{\mathcal{I}}(\tau, z)}{\partial \tau} = \widehat{\mathcal{I}}(\tau, z) - \widehat{\mathbf{S}}(\tau). \tag{2}$$

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Here $z \equiv \mu/\phi(x)$, $\phi(x)$ is the line absorption profile, and the matrix source function $\widehat{\mathbf{S}}(\tau)$ is given by

$$\widehat{\mathbf{S}}(\tau) = \widehat{\mathbf{S}}_*(\tau) + \int_{-\infty}^{+\infty} dz'\, \widehat{\mathbf{G}}(z')\, \widehat{\mathcal{I}}(\tau, z'), \tag{3}$$

$$\widehat{\mathbf{S}}_*(\tau) = \mathrm{diag}(s_I^*(\tau),\, s_Q^*(\tau)), \tag{4}$$

where $s_I^*(\tau)$ and $s_Q^*(\tau)$ are the components of the vector source function of the *scattered* radiation, $\widehat{\mathbf{G}}$ is directly related to the phase matrix of resonance scattering.

The problem with $\widehat{\mathbf{S}}_* \equiv \mathrm{diag}(\sqrt{1-\lambda}, \sqrt{1-0.7W\lambda})$ is called standard (λ is the albedo of single scattering, W is the depolarization parameter). It was analyzed in detail and solved numerically in the works [1, 2] and, also, in [4] where absorption in the continuum was taken into account. In particular, it was shown that the Stokes matrix at $\tau = 0$ for the standard problem can be found from the solution of the matrix generalization of the integral Ambartsumian–Chandrasekhar equation. We denote the solution of this equation by $\widehat{\mathbf{I}}(z)$ (it is the $\widehat{\mathbf{I}}$-matrix).

For the source function matrix $\widehat{\mathbf{S}}(\tau)$ of the problem under consideration it is not difficult to write an integral equation similar to the equation for the analogous scalar source function, when the polarization of the radiation is not taken into account. Application of Sobolev's resolvent method in the case of the atmosphere with an exponential distribution of primary sources (i.e., when $k = 0$ in Eq. (1)) provides the following Stokes matrix for the emergent diffuse radiation:

$$\widehat{\mathcal{I}}(0, z) = \widehat{\mathbf{I}}(z) \left[\frac{z_0^2\, \widehat{\mathbf{I}}^T(z_0)}{z_0 + z} \int_{-\infty}^{\infty} \frac{\widehat{\mathbf{F}}(z')\, dz'}{z_0 + z'} - z_0 \int_0^{\infty} \frac{z'\, \widehat{\mathbf{I}}^T(z')\, \widehat{\mathbf{F}}(z')\, dz'}{(z_0 - z')(z' + z)} \right], \tag{5}$$

where $\widehat{\mathbf{F}}$ is expressed through elements of the matrix $\widehat{\mathbf{G}}$.

In general case, if $k > 0$, it is not difficult to show that the Stokes matrix satisfies the recurrence formula

$$\widehat{\mathcal{I}}_k(0, z) = z_0^2\, \frac{\partial}{\partial z_0}\, \widehat{\mathcal{I}}_{k-1}(0, z). \tag{6}$$

Thus, in the case of primary sources (1), the Stokes matrix of the emergent radiation for any of such sources is expressed in terms of the solution of standard problem $\widehat{\mathcal{I}}(z)$ via the equations (5) and (6).

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Radiative Transfer and Spectra in Stochastic Atmospheres

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Various cosmic objects – stars, active galactic nuclei, accretion discs, etc., suffer the stochastic variations of temperature, large and small scales gas motions, magnetic fields, number densities of atoms and molecules. These stochastic variations give rise to changes of absorption factors, Doppler widths of lines and so on. The existence of numerous reasons for fluctuations lead to a Gaussian distribution of fluctuations. The observed spectra represent quantities averaged over the time and space. The common model explanations do not include the effect of fluctuations. In many cases, the consideration of fluctuations improves the agreement between theoretical explanations and observed values.

1 The radiative transfer equation in stochastic atmosphere

In a stochastic atmosphere the absorption factor has a fluctuating component: $\alpha = \langle\alpha\rangle + \alpha' \equiv \alpha^{(0)} + \alpha'$, $\langle\alpha'\rangle = 0$. The change of radiation intensity along the path s is determined by the equation

$$dI(\mathbf{n}, s) = -[\alpha^{(0)}(s) + \alpha'(s)]\, I(\mathbf{n}, s) ds. \tag{1}$$

The solution of this equation is

$$I(\mathbf{n}, s) = I(\mathbf{n}, 0) \exp\left[-\int\limits_0^s ds'(\alpha^{(0)}(s') + \alpha'(s'))\right] \equiv I(\mathbf{n}, 0) \exp\left(-(\tau^{(0)} + \tau')\right). \tag{2}$$

The average of this expression, adopting for Gaussian probability distribution for fluctuations, gives

$$\langle I(\mathbf{n}, s)\rangle = I(\mathbf{n}, 0) \exp\left[-\left(\tau^{(0)} - \frac{1}{2}\langle\tau'^2\rangle\right)\right] \equiv I(\mathbf{n}, 0) \exp\left(-\tau_{eff}(s)\right). \tag{3}$$

It is seen from this expression that the averaged intensity $\langle I(\mathbf{n}, s)\rangle$ in stochastic medium decreases with the distance weaker than when accounting for the mean absorption factor $\alpha^{(0)}$.

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The radiative transfer equation for $\langle I(\mathbf{n}, s)\rangle$ has the form

$$\frac{d\langle I\rangle}{ds} = -\left[\alpha^{(0)}(s) - \langle\alpha'(s)\tau'(s)\rangle\right]\langle I\rangle + \left[\alpha^{(0)}_{sc}(s) - \langle\alpha'_{sc}(s)\tau'_{sc}(s)\rangle\right] \times \int d\mathbf{n}'\,\kappa(\mathbf{n}\cdot\mathbf{n}')\langle I(\mathbf{n}', s)\rangle + \langle S\rangle. \tag{4}$$

Here $\kappa(\mathbf{n}\cdot\mathbf{n}')$ is the phase function, $\langle S(\mathbf{n}, s)\rangle$ is the averaged source function. Note that $\alpha(s) = \alpha_{sc}(s) + \alpha_{abs}(s)$. The detailed derivation of radiative transfer equations for all Stokes parameters in turbulent magnetized atmosphere is presented in [1].

2 Influence of Doppler width fluctuations on the center of absorption lines

The centers of absorption lines have a Gaussian form, i.e. the broadening is determined by Doppler's mechanism. Fluctuations of the thermal $u_{th} = u^{(0)}_{th} + u'_{th}$ and turbulent $u_{turb} = u^{(0)}_{turb} + u'_{turb}$ velocities give rise to fluctuations of the Doppler width

$$\Delta\lambda_D = \Delta^{(0)}_D + \Delta\lambda'_D, \quad \langle\Delta\lambda'_D\rangle = 0. \tag{5}$$

The level of fluctuations η is determined by the ratio $\eta = |\Delta\lambda'_D|/\Delta\lambda^{(0)}_D$. The averaged value of absorption factor up to $\eta \leq 0.3$ has the form

$$\langle\alpha_\lambda(x)\rangle = \left\langle \frac{\alpha_0}{\Delta\lambda_D}\exp\left[-\left(\frac{\lambda-\lambda_0}{\Delta\lambda_D}\right)^2\right]\right\rangle \simeq \alpha^{(0)}(x)\exp(3x^2\eta^2)\left[(1+\eta^2)\cosh(2\eta x^2) - \eta\sinh(2\eta x^2)\right], \tag{6}$$

where $x = (\lambda - \lambda_0)/\Delta\lambda^{(0)}_D$ and $\alpha^{(0)}(x) = (\alpha_0/\Delta\lambda^{(0)}_D)\exp(-x^2)$. It is seen from this expression that in the center of the line ($x = 0$)

$$\langle\alpha_\lambda(0)\rangle \simeq \alpha^{(0)}(0)\,(1+\eta^2). \tag{7}$$

Thus, the stochastic effect gives rise to additional increase of line's depth. Note that non-LTE models and also synthetic spectra often lead to an increase in the depth of the absorption line. Nevertheless, sometimes this increase is insufficient to explain the observed depth of the line. In these cases the consideration of stochastic effect may help.

3 The influence of fluctuations on spectra in the continuum

The convective and turbulent motions and the magnetic field evolution lead to temperature fluctuations in stellar atmospheres, active galactic nuclei and other cosmic objects. First, we demonstrate the nature of statistical effects by considering two realizations with temperatures $T_0 + T'$ and $T_0 - T'$. The mean value of Planck function (in Wien's limit) is equal to

$$\begin{aligned} \langle B_\lambda(T)\rangle &= \frac{1}{2}\frac{2hc^2}{\lambda^5}\left\{\exp\left[-\left(\frac{h\nu}{k(T_0+T')}\right)\right] + \exp\left[-\left(\frac{h\nu}{k(T_0-T')}\right)\right]\right\} \\ &\simeq B_\lambda(T_0)\cosh\left(\frac{h\nu}{kT_0}\frac{T'}{T_0}\right) \geq B_\lambda(T_0). \end{aligned} \tag{8}$$

Here we accepted $T' \ll T_0$. This simple example demonstrates that the averaged value $\langle B_\lambda(T_0)\rangle$ is larger than $B_\lambda(T_0)$. Analogously one can see that the averaged value of absorption factor $\langle \alpha_\lambda(T)\rangle$ may be either larger than $\alpha_\lambda(T_0)$ or smaller than this value, depending on specific form of function $\alpha_\lambda(T)$. The averaged radiation flux $\langle H_\lambda\rangle$ with allowance for temperature fluctuations is determined as

$$\langle H_\lambda\rangle = 2\pi\int_0^\infty d\langle\tau_\lambda\rangle\int_0^1 d\mu\,\exp\left[-\frac{\langle\tau_\lambda\rangle}{\mu}\right]\frac{\langle\alpha_\lambda(T)B_\lambda(T)\rangle}{\langle\alpha_\lambda(T)\rangle}, \tag{9}$$

where $d\langle\tau_\lambda\rangle = \langle\alpha_\lambda(T)\rangle ds$ determines the averaged optical length. It is interesting that in the Wien limit $\langle H_\lambda\rangle$ can be derived directly from observed spectrum, from the first and second derivatives over λ of the observed spectrum. The fluctuation effects are considered in the papers [2–4].

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