# On the Complex Radiative Transfer in an Optically Finite Homogeneous Atmosphere

T. Viik[1]

E-mail: *tonu.viik@to.ee*

In this paper we consider the classical problem in radiative transfer – the planetary problem – in an isotropically scattering homogeneous optically finite medium where the albedo of single scattering may be defined anywhere in the complex plane.

To solve this problem we use the method of approximating the kernel in the integral equation for the Sobolev resolvent function. This approach allows to define easily determinable auxiliary functions which help us to express almost all the relevant functions of transfer for this problem.

## 1 Statement of the problem

Usually the albedo of single scattering $\lambda$ or $c$ is assumed real in radiative transfer. But when we consider the Laplace transformed time-dependent transport equation $\lambda$ may turn complex. We met another such problem when we tried to determine the photon path-length distribution function in an optically semi-infinite atmosphere. It appeared that the non-linear integral equation for the complex $H$-function is valid even in the complex plane [1]. This interesting fact directed the author to a deeper treatment of the problem and to try to find for such a case the radiation field in general.

In order to solve this problem we used the kernel approximation method first proposed by Krook [2] and later developed by Rybicki [3] and Vainikko et al. [4]. Rybicki approximated the kernel in the integral equation – the exponential integral – for the Sobolev resolvent function $\Phi$ by a Gauss-Legendre sum while Krook and Vainikko et al. used the method of kernel approximation for the integral equation of the source function.

The substitution of the kernel by a Gauss-Legendre sum allows us to solve the obtained approximate equation for the Sobolev resolvent function $\Phi$ [5] exactly while the solution is a weighted sum of exponents. This approach allowed us to define simple auxiliary functions for determining the radiation field.

Here we have chosen to consider the planetary problem in a homogeneous isotropically scattering optically finite atmosphere where the albedo of single scattering is complex

$$\lambda = \lambda_1 + i\lambda_2. \tag{1}$$

[1] Tartu Observatory, Estonia

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We assume that in this case the source function $B$ for a large range of radiative transfer problems in given type of atmospheres can still be described by the well-known Fredholm integral equation

$$B(\tau,\kappa,T)=\frac{1}{2}\lambda\int\limits_0^T E_1(t-\tau)B(t,\kappa,T)dt+\frac{1}{4}\lambda F\exp\left(-\tau/\kappa\right), \tag{2}$$

where $\tau$ is the optical depth, $T$ is the optical thickness of the atmosphere, $\pi F$ is the flux of the incident radiation normal to the plane of stratification, $\kappa$ is the direction cosine of the angle of incidence referred to the outward normal of the atmosphere and the exponential integral is expressed in the form

$$E_n(x)=\int\limits_0^1 \exp(-|x|/s)s^{n-2}ds. \tag{3}$$

Since the Sobolev resolvent function is the regular part of the Green function for Eq. (2) we may immediately write that the solution of Eq. (2) for the planetary case is

$$B(\tau,\kappa,T)=\frac{1}{4}\lambda F\left[\exp\left(-\tau/\kappa\right)+\int\limits_0^\tau \Phi(t,T)\exp\left(-t/\kappa\right)dt\right], \tag{4}$$

where the Sobolev resolvent function satisfies the following Fredholm integral equation [6]

$$\Phi(\tau;T)=\frac{1}{2}\lambda\int\limits_0^T E_1(t-\tau)\,\Phi(t;T)dt+\frac{1}{2}\lambda E_1(\tau). \tag{5}$$

Now the task is all set for the solution of Eq. (5) by approximating the kernel of this equation.

## 2 Solution of the equation for the approximate resolvent function

Eq. (5) is one of the most important equations in the radiative transfer since all the relevant functions of transfer can be expressed through the resolvent function.

We try to solve Eq. (5) by approximating the kernel of it by a sum of exponents

$$E_1(\tau)=\sum_{n=1}^{N} w_n\exp(-\tau/u_n)u_n^{-1}, \tag{6}$$

where $w_n$ are the weights and $u_n$ are the points of a Gauss quadrature rule of the order of $N$ in the interval (0,1). After substitution of this approximation into

Eq. (5) the resulting equation can be solved exactly and the solution is

$$\Phi(\tau, T) = a_1 + b_1\tau + \sum_{i=2}^{N} \left\{ a_i \exp\left(-s_i\tau\right) + b_i \exp\left[s_i(T-\tau)\right] \right\}. \tag{7}$$

If $\lambda_1 \neq 1$ then $a_1 = b_1 = 0$ outside of the summation sign and the summation begins at $i = 1$. This rule applies throughout the paper. The unknown coefficients $s_i$ are the zeros of the equation

$$1 - \lambda \sum_{n=1}^{N} \frac{w_n}{1 - s^2 u_i^2} = 0. \tag{8}$$

The approximate characteristic equation – Eq. (8) – can simply be solved when $\lambda$ is real and positive since we know beforehand in which intervals to search for the zeros. This is not the case when $\lambda$ is complex or negative but if we write Eq. (8) in the polynomial form

$$\sum_{i=1}^{N} c_i s_i^{2i} = 0, \tag{9}$$

we may apply the code DZROOTS from Numerical Recipes [7].

The coefficients $a_i$ and $b_i$ are to be found from linear algebraic systems of equations

$$\begin{aligned} \alpha_1 + \sum_{i=2}^{N} \alpha_i \left[\frac{1}{1 - s_i u_j} + \frac{\exp\left(-s_i T\right)}{1 + s_i u_j}\right] &= u_j^{-1}, \\ \beta_1(T + 2u_j) + \sum_{i=2}^{N} \beta_i \left[\frac{1}{1 - s_i u_j} - \frac{\exp\left(-s_i T\right)}{1 + s_i u_j}\right] &= u_j^{-1}, \quad j = 1, 2, \ldots, N, \end{aligned} \tag{10}$$

while

$$\begin{aligned} &a_i = (\alpha_i + \beta_i)/2, \quad b_i = (\alpha_i - \beta_i)/2, \quad i = 2, \ldots, N; \\ &a_1 = (\alpha_1 - \beta_1 T)/2, \quad b_1 = \beta_1. \end{aligned} \tag{11}$$

This system may be solved, e.g., using algorithms ZGECO and ZGESL from LINPACK.

Thus, the solution of the approximate equation for the Sobolev resolvent function is completed.

# 3 The complex radiation field

Next we define the auxiliary functions $x$ and $y$ as the generalizations of the well-known Ambartsumian–Chandrasekhar functions $X$ and $Y$

$$x(\tau, \kappa, T) = 1 + \int_{\tau}^{T} \Phi(t; T) \exp\left[-(t-\tau)/\kappa\right] dt,$$

$$y(\tau, \kappa, T) = \exp\left(-\tau/\kappa\right) + \int_{0}^{\tau} \Phi(t; T) \exp\left[-(\tau - t)/\kappa\right] dt. \tag{12}$$

By the use of these functions we may express the solution of Eq. (2) [8] as

$$B(\tau, \kappa, T) = \frac{1}{4}\lambda F\{X(\kappa, T) y(\tau, \kappa, T) - Y(\kappa, T)[x(T-\tau) - 1]\}. \tag{13}$$

Evidently, the Ambartsumian–Chandrasekhar functions $X$ and $Y$ are the special cases of $x$ and $y$

$$\begin{aligned} X(\kappa, T) &= x(0, \kappa, T), \\ Y(\kappa, T) &= y(T, \kappa, T). \end{aligned} \tag{14}$$

As the next step we use the well-known definitions for the intensities and find that for the upward moving radiation, i.e. for the intensities towards the smaller optical depths we have

$$\begin{aligned} I(\tau, -\mu, \kappa, T) = \frac{\lambda F}{4}\frac{\kappa}{\mu+\kappa}&\{X(\kappa, T)[x(\tau, \mu, T) + y(\tau, \kappa, T) - 1] \\ &- Y(\kappa, T)[x(T-\tau, \kappa, T) + x(T-\tau, \mu, T) - 1]\} \end{aligned} \tag{15}$$

and for the intensities towards the larger optical depths

$$\begin{aligned} I(\tau, \mu, \kappa, T) = \frac{\lambda F}{4}\frac{\kappa}{\mu-\kappa}&\{X(\kappa, T)[y(\tau, \mu, T) - y(\tau, \kappa, T)] \\ &- Y(\kappa, T)[x(T-\tau, \mu, T) - x(T-\tau, \kappa, T)]\}. \end{aligned} \tag{16}$$

The seeming discontinuity in Eq. (16) may be eliminated by the L'Hopital rule.

# 4 Results

We have performed calculations for different set of atmospheric parameters and we are convinced that at least for the region $-8 \leq \lambda_1 \leq 8$ and $-8 \leq \lambda_2 \leq 8$ our method works well. We checked the results by solving the Ambartsumian–Chandrasekhar differential equations [9] for $X$ and $Y$ functions and coincidence of the results even for the modest $N = 7$ was very good.

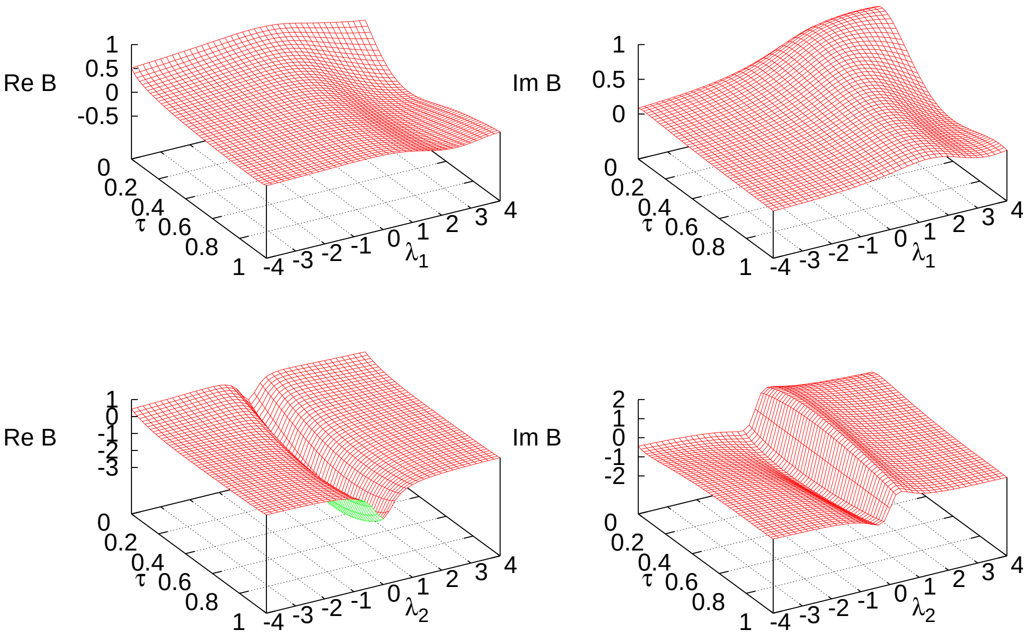


Figure 1: The real and imaginary parts of the source function ($\kappa = 1.0$, $T = 1.0$). Upper panel: $\lambda_2 = 2$, lower panel: $\lambda_1 = 2$.

In Fig. 1 we have presented some results for the source function. One may notice that for the fixed $\lambda_2$ the surfaces – both for the real and imaginary parts – of $B$ are quite smooth while these for the fixed $\lambda_1$ have jumps at $\lambda_2 = 0$. We met a similar behavior when computing the complex $H$ function [1].

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* The color figure is available online in the Proceedings at *http://www.astro.spbu.ru/sobolev100/*.