On Some Applications of General Invariance Relations Reduction Method to Solution of Radiation Transfer Problems

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Foundations of the general invariance relations reduction method are presented outline. A number of solutions of problems of the radiation transfer theory, obtained by the help of this method, is described briefly.

1 Background of the general invariance relations reduction method (GIRRM)

Properties of symmetry and invariance are widely used practically in all fields of people activity [1]. Very often these properties and principles make sense of statements on invariance of some objects, systems, equations, constructions, solutions and so on with respect to sets of actions and operations that form group. However not all properties and principles of invariance can be formulated in the framework of group-theoretical approach [1, 2]. It is necessary to point out a number of the fundamental works in which the concept of the immutability (invariance) solutions of one-dimensional (in the space variables) problems of optics and radiation transfer theory (RTT) under the simplest of the above-mentioned actions and operations are used. These publications include works written by Stokes [3], Ambartsumian (see Refs. in [4]), Chandrasekhar [5], Bellman and Kalaba [6], and Sobolev (see Refs. in [7]). The first principles of invariance in the RTT were formulated by Ambartsumian [4] and Chandrasekhar [5]. Then in 1956 Bellman and Kalaba formulated in a sufficiently abstract way the classical principle of invariant imbedding (PII). More wide interpretation, generalization and application of classical principles of invariance of the RTT were given in a number of works (see Refs. in [2, 8]). The general invariance relations reduction method (GIRRM) was proposed by Rogovtsov [1, 2, 9, 10].

The most important basic GIRRM statements and constructions are the general invariance principle (GIP) and the general invariance relations (GIRs). More narrow formulation of the GIP (in framework of the RTT) was given by Rogovtsov [2, 10]. Most universal formulation of this principle was given in the monograph [1]. By the GIRs we understand consequences of invariance (partial

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invariance) of solutions of problems of the RTT and other mathematical physics problems in respect to above-mentioned actions and operations. The GIRs connect solutions of different or the same type RTT or MP problems. It should be noted that the GIRRM are an heuristic, general and effective method

2 A short list of the results obtained by using the GIRRM

2.1 About rigorous derivation of asymptotic formulas for the plane albedo and spherical albedo for the case of nearly conservative scattering

Using rigorous integral relations and some a priori assumptions Sobolev [7] obtained the following three-term asymptotic formulas:

$$A_{pl}(\mu_1;\omega_0) \sim 1 - 4\sqrt{\frac{1-\omega_0}{3-x_1}}u(\mu_1;1) + b_{Sob}(\mu_1)(1-\omega_0), \quad \omega_0 \to 1, \tag{1}$$

$$A_{sp}(\omega_0) \sim 1 - 4\sqrt{\frac{1-\omega_0}{3-x_1}} + D(3-x_1)^{-1}(1-\omega_0), \quad \omega_0 \to 1. \tag{2}$$

Here $A_{pl}(\mu_1;\omega_0) = 2\int_0^1 \rho^0_{[0,+\infty)}(\mu,\mu_1;\omega_0)\,\mu\,d\mu$ is the plane albedo and $\rho^0_{[0,+\infty)}(\mu,\mu_1;\omega_0)$ is the azimuthally averaged reflection function for a semi-infinite plane-parallel medium (ω_0 is a single scattering albedo), $A_{sp}(\omega_0) = 2\int_0^1 \mu_1 A_{pl}(\mu_1;\omega_0)d\mu_1$ is the spherical albedo, $u(\mu;\omega_0)$ is the function that defines the angular dependence of Milne's problem solution [7, 8], $b_{Sob}(\mu_1) = 15(5-x_2)^{-1}\left(\mu_1^2 - 2\int_0^1 \rho^0_{[0,+\infty)}(\mu,\mu_1;1)\mu^3 d\mu\right) + D(3-x_1)^{-1}u(\mu_1;1)$, where $D = 24\int_0^1 u(\mu;1)\mu^2 d\mu$, $\{x_s\}_{s\in N_0}$ is a sequence of expansion coefficients of the phase function $p(\mu)$ in Fourier series in the system of Legendre polynomials $\{P_s(\mu)\}_{s\in N_0}$ ($p(\mu) = \sum_{s=0}^{+\infty} x_s P_s(\mu); N_0 = \{0,1,2,\dots\}$). Without any a priori assumptions, using the GIRRM, we were strictly obtained Eq. (2) and such asymptotic formula:

$$A_{pl}(\mu_1;\omega_0) \sim 1 - 4\sqrt{\frac{1-\omega_0}{3-x_1}}u(\mu_1;1) + b_{R,B}(\mu_1;\omega_0)(1-w_0), \quad \omega_0 \to 1. \tag{3}$$

The value of $b_{R,B}(\mu_1;\omega_0)$ in Eq. (3) is equal to

$$\begin{aligned} b_{R,B}(\mu_1;\omega_0) = {} & D(3-x_1)^{-1}u(\mu_1;1) \\ & - 15(5-x_2)^{-1}\left[\mu_1^2 - 2\int_0^1 \mu^3 \rho^0_{[0,+\infty)}(\mu,\mu_1;1)d\mu\right] \\ & - 2\int_{-1}^1 d\mu \int_{+0}^{+\infty} \left[\tilde{G}^*_{\infty;0}(\tau,\mu;0,\mu_1;\omega_0) - J(\tau,\mu,\mu_1;\omega_0)\right] d\tau, \end{aligned} \tag{4}$$

$$J(\tau,\mu,\mu_1;\omega_0) = 2\int_0^1 \mu' \tilde{G}^*_{\infty;0}(\tau,\mu,0,-\mu';\omega_0)\rho^0_{[0,+\infty)}(\mu',\mu_1;1)d\mu'.$$

Table 1: Numerical values of the coefficients $b_{Sob}(\mu_1)$ and $b_{R,B;0}(\mu_1)$

μ_1	$g = 0.65$	$g = 0.75$	$g = 0.85$	$g = 0.97$	$g = 0.991$
0.0381347	1.613	2.339	3.901	18.37	60.14
0.238853	3.212	4.652	7.974	41.00	137.3
0.333212	4.080	5.863	9.993	51.15	171.1
0.434867	5.114	7.289	12.34	62.78	209.8
0.539374	6.285	8.894	14.97	75.66	252.6
0.642166	7.545	10.61	17.77	89.32	298.0
0.738751	8.829	12.60	20.60	103.1	343.7
0.824908	10.05	14.02	23.30	116.1	387.0
0.896871	11.14	15.49	26.68	127.6	425.0
0.951494	12.00	16.65	27.54	136.7	455.0
1.0	12.78	17.72	29.26	144.9	482.4

In the formula (4) the function $\tilde{G}^{*}_{\infty;0}(\tau,\mu;\tau',\mu';\omega_0)$ has the meaning of not the main part of the contribution to the integrated over azimuth "volume" Green function [2, 11] of the dimensionless scalar radiative transfer equation (SRTE) for the case of an infinite plane-parallel medium. This part is generated by the subset of the spectrum of the reduced characteristic equation of the SRTT (it corresponds to zero azimuthal harmonic of phase function). The above-mentioned subset does not contain only the minimum in modulus eigenvalues. The asymptotic formulas (1) and (3) differ in shape (their third members have different forms). Nonetheless theoretical analysis and the series of numerical experiments showed that there are no differences (within the limits of calculation errors for sufficiently small values of q $(q = 1-\omega_0)$ between the coefficients $b_{Sob}(\mu_1)$ and $b_{R,B}(\mu_1;\omega_0)$.

Remark 1. *The coefficient $b_{R,B}(\mu_1;\omega_0)$ can be represented in form of the following series:*

$$b_{R,B}(\mu_1;\omega_0) = \sum_{l=0}^{+\infty} b_{R,B;l}(\mu_1)(1-\omega_0)^l. \qquad (5)$$

This series is convergent point-wise and uniformly on [-1,1] for sufficiently small values of q.

Remark 2. *Using the GIRRM the effective analytical and numerical algorithms for finding all the quantities in the asymptotic formulas* (2) *and* (3) *for any phase functions are developed.*

In Table 1 a number of numerical values of the coefficients $b_{Sob}(\mu_1)$ and $b_{R,B;0}(\mu_1)$ are given for the case of Henyey-Greenstein's phase function $\chi(\mu;g)$ [7, 8]. From the above-said and Table 1, it follows that the assumptions [7], which V.V. Sobolev used in the derivation of the formulas (1) and (2), are correct for situations considered.

2.2 The correct methods of derivation of multi-term asymptotics for the case of plane-parallel media

The GIRRM allows to derive asymptotic formulas for azimuthally averaged reflection and transmission coefficients [7] for the case of plane-parallel optically thick media without using a priori assumptions about their structures. It should be noted that it is necessary to take into account the implicit contribution of the entire spectrum of the characteristic equation (CE) of the SRTE in the above-mentioned coefficients in the process of rigorous derivation of these asymptotics. In particular, all of the elements (they belong to the spectrum of the CE), which do not coincide with minimal in modulus elements of the same spectrum, contribute some terms of the order of $(1-\omega_0)$ (if $\omega_0 \to 1$) to asymptotics of these coefficients. Using some constructions of the GIRRM, the principle of reciprocity [11] and the analytical representations (see [12, 13]) for the "volume" Green function of the SRTE for a infinite plane-parallel medium, we have proved the faithfulness of Sobolev's a priori assumptions and asymptotic formulas [7] for the above-mentioned coefficients for the cases of semi-infinite media and layers of a large optical thickness τ_0. In [14] multi-term formulas for the reflection and transmission coefficients when $\tau_0 \to \infty$ were first obtained in implicit form. These asymptotics were found by using Case's method. Then the methods of finding of multi-terms asymptotics of various radiative characteristics were proposed in [15, 16, 17]. The most effective algorithm for deriving of such asymptotics was described in [15]. This algorithm was based on the constructive ideas of the GIRRM. To illustrate capabilities of this algorithm we write down only some relations from [15]. Consider a macroscopically homogeneous and local isotropic plane-parallel turbid layer of an optical thickness τ_0. Then using standard constructions of the GIRRM [1, 2] the following GIRs:

$$\rho^0(|\mu|,\xi;\omega_0,\tau_0) = g_1^0(|\mu|,\xi;\omega_0,\tau_0) + \int_0^1 K(|\mu|,\mu'';\omega_0,\tau_0)\rho^0(\mu'',\xi;\omega_0,\tau_0)d\mu'', \quad (6)$$

$$\sigma^0(|\mu|,\xi;\omega_0,\tau_0) = g_2^0(|\mu|,\xi;\omega_0,\tau_0) + \int_0^1 K(|\mu|,\mu'';\omega_0,\tau_0)\sigma^0(\mu'',\xi;\omega_0,\tau_0)d\mu'', \quad (7)$$

$$(|\mu|,\xi \in [0,1],\ \omega_0 \in [0,1],\ \tau_0 \in (0,+\infty))$$

were obtained in [15]. In GIRs (6) and (7) the functions $\rho^0(|\mu|,\xi;\omega_0,\tau_0)$ and $\sigma^0(|\mu|,\xi;\omega_0,\tau_0)$ are the azimuthally averaged reflection and transmission coefficients [7, 8] of a layer correspondingly. The function $K(|\mu|,\mu'';\omega_0,\tau_0)$ is defined by the relation.

$$K(|\mu|,\mu'';\omega_0,\tau_0) = \mu'' \int_0^1 \mu' \tilde{G}_{[0,+\infty)}(0,-|\mu|;\tau_0,\mu';\omega_0)\tilde{G}_{[0,+\infty)}(0,-\mu';\tau_0,\mu'';\omega_0)d\mu'. \quad (8)$$

Here function $\tilde{G}_{[0,+\infty)}(\tau,\mu;\tau',\mu';\omega_0)$ is the "volume" Green function of the dimensionless SRTE for the case of a semi-infinite plane-parallel medium which comprises the "sources" $\delta(\tau-\tau')\delta(\mu-\mu')$ $(\tau'>0)$. In turn the functions $g_1^0(|\mu|,\xi;\omega_0,\tau_0)$, $g_2^0(|\mu|,\xi;\omega_0,\tau_0)$ can be expressed in terms of values of this "volume" Green function (see [15]).

For example, using the principle of reciprocity [11], the representations for the Green function $\tilde{G}_{[0,+\infty)}(\tau,\mu;\tau',\mu';\omega_0)$ [12, 13], K-integral of the SRTE [7] and GIRs (6), (7) the following asymptotics:

$$
\begin{aligned}
\sigma^0(|\mu|,\xi;1,\tau_0) &= Q(|\mu|,\xi;\tau_0) + (2u(|\mu|;1) + h_2(|\mu|;\tau_0))\gamma_1(\tau_0,x_1) \\
&\quad \times \left\{ \int_0^1 \mu'^2 \rho^0_{[0,+\infty)}(\mu',\xi;1)d\mu' + h_1(\xi;\tau_0) + \gamma_2(\tau_0,\xi,x_1) \right\} \\
&\quad + O(\tau_0^{-1}\exp(-2k_2\tau_0)), \quad \tau_0 \to +\infty, \\
\gamma_1(\tau_0,x_1) &= \left[(1-\frac{x_1}{3})\tau_0 + 4\int_0^1 \mu'^2 u(\mu';1)d\mu' + h(\tau_0) \right]^{-1}, \\
\gamma_2(\tau_0,\xi,x_1) &= 2^{-1}\xi\Big(1-\exp\Big(-\frac{\tau_0}{\xi}\Big)\Big) - 2^{-1}\Big(1-\frac{x_1}{3}\Big)\tau_0\exp\Big(-\frac{\tau_0}{\xi}\Big),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\int_0^1 \mu\sigma^0(\mu,\xi;1,\tau_0)d\mu = \gamma_1(\tau_0,x_1) &\left\{ \int_0^1 \mu'^2 \rho^0_{[0,+\infty)}(\mu',\xi;1)d\mu' \right. \\
&\left. + h_1(\xi;\tau_0) + \gamma_2(\tau_0,\xi,x_1) \right\} \\
&+ O(\tau_0^{-2}\exp(-2k_2\tau_0)), \qquad \tau_0 \to +\infty,
\end{aligned}
\tag{10}
$$

were obtained in [15]. The functions $Q(|\mu|,\xi;\tau_0)$, $h(\tau_0)$, $h_1(\xi;\tau_0)$, $h_2(|\mu|;\tau_0)$ are expressed explicitly in terms of the functions $u(|\mu|;1)$, $\rho^0_{[0,+\infty)}(|\mu|,\xi;1)$, $\tilde{G}^*_{\infty;0}(\tau,\mu;\tau',\mu';\omega_0)$. In addition there are asymptotics $h(|\mu|;\tau_0) = O(\exp(-k_2\tau_0))$, $h_1(\xi;\tau_0) = O(\exp(-k_2\tau_0))$, $h_2(|\mu|;\tau_0) = O(\exp(-k_2\tau_0))$, $\tau_0 \to +\infty$. In Eqs. (9), (10) under symbol k_2 it should be understood the second non-negative root of the reduced characteristic equation of the SRTE (if it exists). If a root does not exist under the symbol k_2 it is necessary to understand the positive number $(1-\varepsilon)$, where ε is a small enough positive number. Eq. (9) is a generalization of asymptotics for $\sigma^0(|\mu|,\xi;1,\tau_0)$ obtained by Sobolev [7].

2.3 Constructive theory of scalar characteristic equations of the radiative transfer theory

The constructive theory of scalar characteristic equations of the RTT was suggested in [12, 13, 18]. The construction of solutions of these equations in analytic form can be reduced to finding solutions of infinite tridiagonal systems of linear algebraic equations. Effective analytical and numerical algorithms for finding discrete spectra, eigenfunctions and normalizing constants for reduced scalar characteristic equations of the SRTE was described in above-mentioned works. New two-term recursion formulas and analytic representations for solutions of infinite tridiagonal systems of linear algebraic equations were suggested in [13]. In addition, Rogovtsov obtained a general analytic expression for the "volume"

Green function of a two-dimensional (with respect to the angular variables) integro-differential equation of the radiative transfer for the case in which the phase function satisfies the Hölder condition on [-1, 1].

2.4 Effective algorithms for finding the reflection function, plane and spherical albedo for any phase function

Properties of invariance are used in the RTT in developing the effective algorithms for finding the reflection function, plane and spherical albedo. Point out two algorithms, in which these properties are used in an explicit form. The first algorithm is based on the use of Ambartsumian's non-linear integral equations for the reflection function and its azimuthal harmonics. The second algorithm was developed through the use of Fredholm special integral equations. The nonlinear integral above-mentioned equations were obtained by Ambartsumian by using the principle of invariance which he formulated in 1943 (see Refs. in [4]). Special Fredholm equations were found through the use of rigorous mathematical considerations or some properties of invariance in a number of papers (see, for example, [2, 12, 19, 20, 21] and references therein). The first algorithm was used, in particular, in [22]. The second algorithm is actually used in [2, 12]. It should be noted that the correct application of both algorithms requires the use of additional information about solutions of other problems of the RTT. For example, the quantities describing the deep regime of the radiation intensity in a semi-infinite medium and the Sobolev–van de Hulst relation [7, 8, 22] was used in the first algorithm [22] as an additional information in the construction of a sustainable iterative algorithm for solving nonlinear scalar Ambartsumian's integral equations. Previously it is necessary to find "volume" Green function of the SRTE for the case of an infinite plane-parallel turbid medium if special Fredholm integral equations are taken as the initial equations when finding of the reflection function. Before developing effective analytical and numerical algorithms for finding the above-mentioned Green function for cases of arbitrary phase functions it was practically impossible to use this kind of equations. Such algorithms were constructed and effectively used in [2, 12, 13]. These algorithms can be applied for the cases of sharply anisotropic phase functions. To illustrate the effectiveness of the algorithms developed in [2, 12, 13] we give below Table 2 for the quantities $A_{pl}(\mu_1; \omega_0)$, $A_{sp}(\omega_0)$ for the case of the phase function $\chi(\mu; g)$.

2.5 Exact expressions, asymptotic formulas, inequalities and asymptotic inequalities for the average characteristics of radiative fields in turbid media of different configurations

Different GIRs can be used for finding the integral invariants of the stationary and non-stationary SRTE. Moreover a number of average characteristics of radiative fields in turbid media of different configurations can be found using the GIRs. In the most simple form such results were obtained by Rogovtsov (see [1, 2, 13] and

Table 2: Values for plane and spherical albedo $A_{pl}(\mu_1;\omega_0)$, $A_{sp}(\omega_0)$ $(g = 0.993)$

μ_1	$\omega_0 = 0.99$	0.993	0.997	0.999	0.9999	0.99999
0.844195×10^{-2}	0.6995	0.7326	0.7996	0.8668	0.9504	0.9834
0.381347×10^{-1}	0.5356	0.5833	0.6836	0.7879	0.9206	0.9734
0.880185×10^{-1}	0.4097	0.4643	0.5860	0.7191	0.8939	0.9643
0.155914	0.3135	0.3670	0.5027	0.6575	0.8691	0.9558
0.238853	0.2407	0.2948	0.4314	0.6016	0.8453	0.9476
0.434867	0.1456	0.1906	0.3202	0.5057	0.8010	0.9318
0.642166	0.09391	0.1294	0.2442	0.4312	0.7622	0.9174
0.738751	0.07811	0.1097	0.2170	0.4020	0.7456	0.9111
0.896871	0.05896	0.08970	0.1803	0.3598	0.7200	0.9011
1.0	0.04964	0.07264	0.1605	0.3353	0.7040	0.8948
$A_{sp}(\omega_0)$	0.1079	0.1431	0.2542	0.4351	0.7612	0.9167

Refs. in therein) for the cases turbid media having forms of layer, sphere, infinite circular cylinder and regular polyhedral. In these works the average duration of the luminescence and radiative fluxes were required quantities. In turn the asymptotic inequalities for the mean intensity of the radiation, the average number of scattering of a photon, the average density of radiation, radiative fluxes and spherical albedo were found by Rogovtsov, Karpuk and Samson (corresponding Refs. are given in [2, 13]). These authors considered the process of radiative transfer in turbid media that have the forms of layer, sphere, finite and infinite circular cylinders, spheroids, spherical shell and non-concavity body bounded by a smooth boundary. In some of the above-mentioned publications the presence of underlying surfaces was allowed.

2.6 On the asymptotic expressions for the Green functions of the SRTE when turbid medium contains mono-directional point or line sources

2.6.1. Let turbid “medium” $\tilde{V}$ be a macroscopically homogeneous or two-layer non-conservative semi-infinite “medium” $\tilde{V}_{[0,\infty)}$, which is irradiated by an infinitely narrow mono-directional beam of radiation or contains near its border $\tilde{S}_{[0,\infty)}$ a point mono-directional source (see Figs. 1a, 1b).

Then the intensity of the radiation (or Green functions) at an optical depth τ_0 (when $\tau_0 \to +\infty$) at any observation point P can be represented in a simple analytical form (see [23]). In addition, the principle terms of asymptotic formulas are expressed in terms of elementary functions and solutions of special BVPs for the case of a plane-parallel anisotropic absorbing semi-infinite turbid “medium”.

Remark 3. *Under the above-mentioned assumptions the forms of relative intensities for deep regime behaviors tend asymptotically to each other when a semi-infinite turbid medium is irradiated by an infinitely wide mono-directional beam of radiation or infinitely narrow mono-directional beams of radiation.*

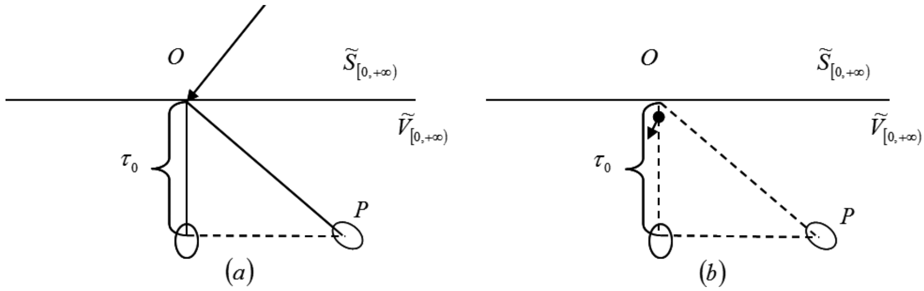

Figure 1: Geometries of problems for the cases of external and internal sources of radiation.

Remark 4. *Let the observation point P be at a large optical depth τ_0 and the shortest optical distance from point P to a perpendicular to $\tilde{S}_{[0,\infty)}$ (it passes through the point of incidence of external radiation) is equal τ_1. Then for the case of a macroscopically homogeneous turbid medium the principle term of the asymptotics of "volume" Green function of the dimensionless SRTE will be in form [24]*

$$\tilde{G}_{[0,+\infty)}\big(\vec{\tau},\vec{\Omega};\vec{0},\vec{\Omega}_1;\omega_0\big) \sim \frac{k_1 \exp(-k_1\tau_0)}{2\pi^2\tau_0}\, i(\mu;\omega_0)\, u(\mu_1;\omega_0),$$
$$\omega_0 \in (0,1), \quad \tau_0 \to +\infty, \quad (\tau_1/\sqrt{\tau_0}) \to 0 \tag{11}$$

Here functions $i(\mu;\omega_0)$ and $u(\mu_1;\omega_0)$ are the classic functions of the SRTT [7, 8]; k is the smallest positive element of the discrete spectrum of characteristic equation of the SRTT [7, 8, 13].

2.6.2. Let $\tilde{V}_{[0,\tau_0]}$ be a macroscopically uniform non-conservative scattering "layer" of an optical thickness τ_0 which is irradiated by a mono-directional infinitely narrow beam of radiation (see Fig. 2). Then with the help of the GIRRM the principle term of asymptotics of the "surface" Green function [11] of the dimensionless the SRTE for any position of observation point P, which is on the second boundary $\tilde{S}_2$ of the layer $\tilde{V}_{[0,\tau_0]}$, can be found. Here τ_0 tends to $+\infty$.

Remark 5. *Let the shortest optical distance from an observation point P to the perpendicular to $\tilde{S}_1$ which passes through the point of incidence of external*

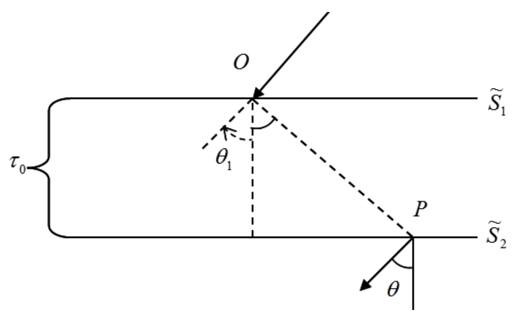

Figure 2: Geometry of problem for the case of the layer irradiated by the external beam of radiation.

radiation be equal to τ_1. Then the principle term of the asymptotics of the "surface" Green function $\tilde{G}_{\tilde{S}}(\tau, \vec{\Omega}; \vec{0}, \vec{\Omega}_1; \omega_0; \tilde{V}_{[0,\tau_0]})$ of the SRTE can be presented in the form [24]

$$\tilde{G}_{\tilde{S}}(\vec{\tau}, \vec{\Omega}; \vec{0}, \vec{\Omega}_1; \omega_0; \tilde{V}_{[0,\tau_0]}) \sim \mu_1 \frac{M k_1 \exp(-k_1\tau_0)}{2\pi^2\tau_0} u(\mu; \omega_0)\, u(\mu_1; \omega_0),$$
$$\tau_0 \to +\infty, \quad \frac{\tau_1}{\sqrt{\tau_0}} \to 0, \quad \omega_0 \in (0,1), \tag{12}$$
$$M = 2\int_{-1}^{1} \mu\, i^2(\mu; \omega_0) d\mu, \quad \mu = \cos\theta, \quad \mu_1 = \cos\theta_1.$$

2.6.3. Let $\tilde{V}$ be a non-conservative scattering "medium" having the shape of a sphere, the center of which is the point mono-directional "source" $\delta(\vec{\tau})\delta(\vec{\Omega} - \vec{\Omega}_1)$. In addition, the optical radius of $\tilde{V}$ is equal to τ_0. Then the asymptotic formula (see Refs. in [1, 2])

$$\tilde{G}(\vec{\tau}, \vec{\Omega}; \vec{0}, \vec{\Omega}_1; \omega_0; \tilde{V}) \sim \frac{k_1 \exp(-k_1\tau_0)}{2\pi^2\tau_0} u((\vec{n}\cdot\vec{\Omega}))\, i((\vec{n}\cdot\vec{\Omega}_1)), \tag{13}$$
$$\tau_0 \to +\infty, \quad ((\vec{n}\cdot\vec{\Omega}) \geq \varepsilon > 0)$$

holds. Here $\vec{n}$ is the unit dimensionless external normal to the boundary $\tilde{S}$ of the "medium" $\tilde{V}$ in a observation point (it is specified by an optical radius-vector $\vec{\tau}$) which lies at this boundary.

2.6.4. Let $\tilde{V}$ be a non-conservative scattering "medium" which has the shape of an infinite circular cylinder and contains (on the axis of symmetry) a linear mono-directional "source" $\delta(\vec{x})\delta(\vec{y})\delta(\vec{\Omega} - \vec{\Omega}_1)$ (see Fig. 3). Then the asymptotic formula (see Refs. in [1, 2])

$$\int_{-\infty}^{+\infty} \tilde{G}(\vec{\tau}_p, \vec{\Omega}; \tilde{z}\vec{e}_3, \vec{\Omega}_1; \omega_0; \tilde{V}) d\tilde{z} \sim \frac{1}{\pi}\sqrt{\frac{k_1}{2\pi\tau_0}} \exp(-k_1\tau_0)\, u((\vec{n}\cdot\vec{\Omega})\, i(\vec{n}\cdot\vec{\Omega}_1)), \tag{14}$$
$$\tau_0 \to +\infty \quad ((\vec{n}\cdot\vec{\Omega}) \geq \varepsilon > 0), \quad \omega_0 \in (0,1)$$

holds. Here $\vec{e}_3$ is the unit dimensionless vector which defines the direction of $\tilde{Z}$-axis of a dimensionless Cartesian right rectangular coordinate system $O\tilde{X}\tilde{Y}\tilde{Z}$ (the axis $\tilde{Z}$ coincides with symmetry axis of this cylinder), $\vec{\tau}_p$ specifies an observation point P, which is on the boundary of the cylinder.

2.6.5. Let V be a non-conservative scattering medium, which has a disk shape (see Fig. 4). We will assume that the local optical characteristics of V can depend only on the depth z in a Cartesian right rectangular coordinate system $OXYZ$. Assume that a plane OXY is parallel to the plane parts of the boundary of the disk V and the point O is situated on the axis of symmetry of the disk (this point should be situated inside the disk). Let the disk V contain a point isotropic "source" $\delta(\vec{r})$, which is located at the point O.

Using the GIRRM an asymptotic formula for the "volume" Green function was obtained when the observation point P is situated on the lateral boundary

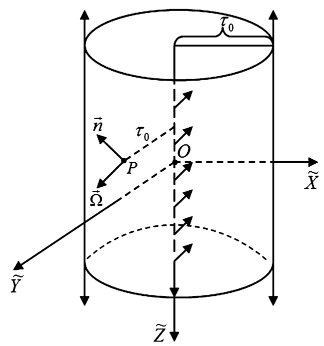

Figure 3: Geometry of the problem for the case of the infinite circular cylinder

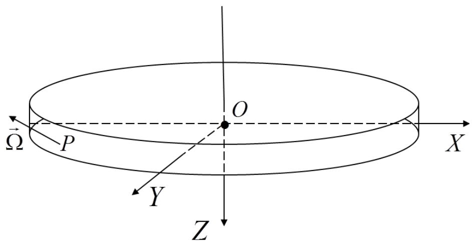

Figure 4: Geometry of the problem for the case of the disk

of the disc and the radius R of the disk tends to $+\infty$. This asymptotics has the following form (see Refs. in [1, 2]):

$$\tilde{G}(\vec{r}_p, \vec{\Omega}; \vec{0}; V) \sim \frac{c_1}{\sqrt{R}} \exp(-k^* R)\, B(z; \vec{\Omega}), \quad R \sup_{z\in[a,b]}\{\alpha(z)\} \to +\infty. \tag{15}$$

Here k^* is the smallest positive root of the non-classical characteristic equation of the SRTT, the constant c_1 is expressed through the first eigenvalue and the corresponding eigenfunction of this equation, $\alpha(z)$ is an attenuation coefficient, the function $B(z; \vec{\Omega})$ is expressed through solutions of one-dimensional and two-dimensional (in space variable) BVPs (the initial BVP is three-dimensional).

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References

1. *N.N. Rogovtsov*, Properties and Principles of Invariance. Application to Solving of Problems of Mathematical Physics, Part 1. Minsk: BGPA, 1999.

2. *N.N. Rogovtsov*, General Invariance Relations Reduction Method and Its Applications to Solutions of Radiative Transfer Problems for Turbid Media of Various Configurations, in Light Scattering Reviews, vol. 5. Ed. A.A. Kokhanovsky. Chichester: Springer-Praxis, 2010, pp. 243-327.

3. *G.G. Stokes*, Proc. Roy. Soc. Lond., **11**, 545, 1862.

4. *V.A. Ambartsumian*, Nauchnye trudy, Vol.1. Yerevan: Publ. Co. Acad. Sci. Arm. SSR, 1960.

5. *S. Chandrasekhar*, Radiative Transfer. London: Oxford University Press, 1950.

6. *R. Bellman, R. Kalaba*, Proc. Nat. Acad. Sci., **42**, 629, 1956.

7. *V.V. Sobolev*, Light Scattering in Planetary Atmospheres. New York: Pergamon Press, 1975.

8. *E.G. Yanovitskij*, Light Scattering in an Inhomogeneous Atmosphere. New York: Springer-Verlag, 1997.

9. *N.N. Rogovtsov*, J. Appl. Spectrosc., **34**, 241, 1981; *ibid*, **35**, 1354, 1981.

10. *N.N. Rogovtsov*, Dokl. Akad. Nauk BSSR, **25**, 420, 1981.

11. *K.M. Case, P.F. Zweifel*, Linear Transport Theory. Massachusetts: Addison-Wesley Publ. Co., 1967.

12. *N.N. Rogovtsov, F.N. Borovik*, The Characteristic Equations of Radiative Transfer Theory, in Light Scattering Reviews, vol. 4 Ed. A.A. Kokhanovsky. Chichester: Springer-Praxis, 2009, pp. 347–429.

13. *N.N. Rogovtsov*, Diff. Equat., **51**, 268, 2015; *ibid*, **51**, 661, 2015.

14. *N.V. Konovalov*, Preprint No. 65, Moscow: Inst. Appl. Math., 1974.

15. *N.N. Rogovtsov*, Teoreticheskaya i Prikladnaya Mekhanika (Minsk), **22**, 72, 2007.

16. *N.N. Rogovtsov, A.M. Samson*, Astrophys., **23**, 468, 1985.

17. *N.N. Rogovtsov*, Dokl. Akad. Nauk BSSR, **41**, 52, 1997.

18. *N.N. Rogovtsov*, in Boundary-Value Problems, Special Functions and Fraction Calculus. Ed. A.A. Kilbas. Minsk: BGU, 1996, pp. 305–312.

19. *N.N. Rogovtsov*, Izv. Atmos. Ocean. Phys., **16**, 160, 1980.

20. *N.N. Rogovtsov, A.M. Samson*, J. Appl. Spectrosc., **25**, 1164, 1976.

21. *H. Domke*, J. Quant. Spectrosc. Rad. Transf., **16**, 973, 1976.

22. *M.I. Mishchenko, J.M. Dlugach, E.G. Yanovitskij, N.T. Zakharova*, J. Quant. Spectrosc. Rad. Transf., **63**, 409, 1999.

23. *N.N. Rogovtsov*, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okean., **26**, 1082, 1990.

24. *N.N. Rogovtsov*, Astrophys., **29**, 781, 1988.