On the Linear Properties of the Nonlinear Problem of Radiative Transfer

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We address the nonlinear problem of reflection/transmission of radiation from an anisotropic scattering/absorbing one-dimensional medium of finite geometrical thickness, when both of its boundaries are illuminated by intense monochromatic radiative beams. The new conceptual element of so-called "linear images" is noteworthy, which admits a probabilistic interpretation. The solution of nonlinear problem of reflection/transmission of radiation is reduced to a linear combination of linear images. They describe the reflectivity and transmittance of the medium for a single photon or their beam of unit intensity, incident on one of the boundaries of the layer, when the medium in real regime is still under the bilateral illumination by external exciting radiation of arbitrary intensity. To determine the linear images, we exploit three well known methods: (i) adding of layers, (ii) its limiting form described by differential equations of invariant imbedding, and (iii) a transition to the so-called functional equations of Ambartsumyan's "complete invariance".

1 Introduction

In linear problem of transfer of radiation energy, the resulting characteristics of the radiation field are formed in the process of multiple interactions of radiation with matter, when the physical properties of the medium are assumed to be unchanged. The very complexity of nonlinear problem, in contrast, is the functional dependence of the scattering/absorbing properties of each elementary volume $\Delta \to 0$ on the intensity of radiation incident on it from all sides. The characteristics of the diffusing in medium radiation field and the physical state of the medium itself are forming each other reciprocally, in a self-consistent manner.

It is well known that in the linear case, the solution of the problem of reflection-transmission (PRT) of radiation, i.e. seeking the intensities $u_L^{\pm}(x,y)$ of emerging radiation from the right "+" and left "−" boundaries of the anisotropic medium (of finite geometrical thickness L), which is illuminated from both boundaries simultaneously by intense radiation beams with intensities x and y, respectively, is reduced to a simple linear combination of the solutions of the two separate problems of its unilateral illumination (from left x, and from right y, separately):

$$u_L^{+}(x,y) = q^{+}\,x + r^{+}\,y, \tag{1}$$

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$$u_L^-(x,y) = r^- x + q^- y, \tag{2}$$

where r^- and q^+ are the coefficients of reflection and transmission of an anisotropic medium of geometric thickness L, for a “single quantum”, or their “beam of unit intensity”, incident from its left boundary, while r^+ and q^- are their counterparts related to the right boundary. These coefficients can be readily interpreted as the probability densities of reflection and transmission of a single photon incident on one of the two boundaries of medium. In nonlinear case [1]–[5], the relations analogous to Eqs. (1)–(2) do not hold. The relationship of these two problems is now implemented (instead of Eqs. (1)–(2)) by Cauchy problems [1]–[5]. Moreover, it is obvious that in the nonlinear case, it makes no sense at all to operate with such concepts as “single photon”, or their “beam of unit intensity”, and the use of probabilistic interpretation of transference phenomena, which though are very efficient tools in the linear problems. This obstacle, in nonlinear problems of bilateral illumination of medium, still prevents to explore only the solution of equations for particular PRT of unilateral illumination of medium (such as seeking the variables $r^\pm$ and $q^\pm$ [6, 7] of the linear case). Therefore, the exact methods of determining the field of radiation emerging from the medium, such as: (i) adding of layers, (ii) its limiting form, described by differential equations of invariant imbedding, and (iii) the so-called functional equations of Ambartsumyan's “complete invariance” (ACI) [4, 5], are compelled here to apply directly to the functions $u_L^\pm(x,y)$ of bilateral illumination of medium, which significantly complicates their determination.

A major goal of this report is to simplify further the methods of nonlinear PRT by revealing and exploring some new functions of so-called “linear images” of the solution of PRT. It is noteworthy that the solution in quest of nonlinear PRT is expressed in terms of newly introduced functions explicitly, just as in the linear case, through a simple linear combination of the solutions of more particular PRT of unilateral illumination of medium. We show that the introduction of these linear images allows to handle effectively a random walk of a single quantum or their unit beam. Moreover, this ensures an application of Sobolev's probabilistic interpretation [8] of linear transfer problems, in nonlinear case too, as simple as in the linear case. For a determination of linear images, as a consequence of the systematic application of the principle of invariance [1] and [4]–[6], we explore in unified way the analogues of described above all three methods of solutions of PRT.

2 The linear images of nonlinear PRT

The idea of introduction of linear images is closely related to one observation of Ambartsumian [9, 2] that inevitably a translucence of medium occurs at high intensities of external radiation exiting it, which is due to the transition of essential fraction of atoms from the ground state to an excited. As a result, the proportion of absorbing neutral atoms in the medium decreases and a stationary regime was established in the excited medium with a new, changed, optical thickness. The original problem becomes linear with respect to new values of the optical thickness, unknown in advance. This is just a physical meaning of the approach entitled "method of self-consistent optical depths", and further used very effectively by [10]. Following Ambartsumian, let us trace the path of a single quantum, randomly walking in an anisotropic scattering/absorbing medium, when certain steady state conditions are established in it. This constant level of an excitation of medium is maintained (stationary regime) during the whole process of random walk of a quantum. It means that an arbitrary chosen quantum just "lives" in a linear medium during this entire time. If x number of photons are incident on medium from the left, and y – from the right, then their total output, as in linear case, can be given by the relations analogous to Eqs. (1)–(2), with the only difference that the functions, $R^{\pm}_{L}(x,y)$ and $T^{\pm}_{L}(x,y)$, of the described above linear images of solution of original PRT, are dependent on the total number of photons (x,y), because of nonlinearity of the problem:

$$u^{+}_{L}(x,y) = T^{+}_{L}(x,y)\,x + R^{+}_{L}(x,y)\,y, \tag{3}$$

$$u^{-}_{L}(x,y) = T^{-}_{L}(x,y)\,y + R^{-}_{L}(x,y)\,x. \tag{4}$$

The functions $R^{\pm}_{L}(x,y)$ and $T^{\pm}_{L}(x,y)$ are the above-mentioned linear images of the solution of original PRT. They are, respectively, the probability densities of reflection and transmission of a single photon or their unit beam incident on the medium through from one of its two boundaries. Although these functions describe the behavior of a single quantum or their unit beam, but because of nonlinearity of the problem nevertheless depend upon the intensities (x,y) of entering medium radiation, due to which the acting level of an excitation of medium has been set. In asymptotic limit of weak fields $x + y \leq \delta^{\pm}$, these functions apparently become constants, which are the solutions of a linear problem, where $\delta^{\pm}$ is the asymptotic threshold of incident single quantum from the left and right, respectively.

3 Relations of the adding of layers for the linear images

As a first method for determining the linear images, let us employ a general method of adding of layers in the nonlinear problems of transfer [1, 2, 5]. Suppose the anisotropic one-dimensional medium of geometrical thickness of B is adjoined from the right to a similar medium of thickness A. Thereby the composite slab of finite thickness $A + B$ is illuminated from the left and right boundaries by radiation of intensities x and y, respectively. Required to determine the intensity of the radiation $u^{\pm}_{A+B}(x,y)$ emerging from this composite slab, when the solutions of similar problems for its both sub-layers, $u^{\pm}_{A}(x,y)$, $u^{\pm}_{B}(x,y)$, are previously known. From Eqs. (3)–(4), it is seen that the problem is reduced to determination of the linear images $R^{\pm}_{A+B}(x,y)$ and $T^{\pm}_{A+B}(x,y)$ by means of known linear images $R^{\pm}_{A}(x,y)$, $T^{\pm}_{A}(x,y)$ and $R^{\pm}_{B}(x,y)$, $T^{\pm}_{B}(x,y)$. From the formulas of the nonlinear addition of layers [5], by virtue of Eqs. (3)–(4), we obtain

$$T^{+}_{A+B}(x,y) = T^{+}_{B}(p,y)\ p_{+}, \tag{5}$$

$$R^{+}_{A+B}(x,y) = R^{+}_{B}(p,y) + T^{+}_{B}(p,y)\ p_{-}, \tag{6}$$

$$R^{-}_{A+B}(x,y) = R^{-}_{A}(x,s) + T^{-}_{A}(x,s)\ s_{+}, \tag{7}$$

$$T^{-}_{A+B}(x,y) = T^{-}_{A}(x,s)\ s_{-}, \tag{8}$$

where the four auxiliary functions can be obtained exploring the explicit relations

$$p_{+} = \frac{T^{+}_{A}(x,s)}{1 - R^{+}_{A}(x,s)\ R^{-}_{B}(p,y)}, \quad s_{+} = \frac{R^{-}_{B}(p,y)\ T^{+}_{A}(x,s)}{1 - R^{-}_{B}(p,y)\ R^{+}_{A}(x,s)}, \tag{9}$$

$$p_{-} = \frac{R^{+}_{A}(x,s)\ T^{-}_{B}(p,y)}{1 - R^{+}_{A}(x,s)\ R^{-}_{B}(p,y)}, \quad s_{-} = \frac{T^{-}_{B}(p,y)}{1 - R^{-}_{B}(p,y)\ R^{+}_{A}(x,s)}. \tag{10}$$

The unknowns, p and s, can be found from the system

$$\begin{cases} p = T^{+}_{A}(x,s)\ x + R^{+}_{A}(x,s)\ s, \\ s = T^{-}_{B}(p,y)\ y + R^{-}_{B}(p,y)\ p, \end{cases} \tag{11}$$

or writing them in the form of separate equations

$$p = g^{+}(x,y;p,s) + K^{+}(x,y;p,s)\ p, \quad s = g^{-}(x,y;p,s) + K^{-}(x,y;p,s)\ s. \tag{12}$$

Here the proper kernels and free terms are defined by

$$\begin{aligned} K^{+}(x,y;p,s) &\equiv R^{+}_{A}(x,s)\ R^{-}_{B}(p,y), \\ K^{-}(x,y;p,s) &\equiv R^{-}_{B}(p,y)\ R^{+}_{A}(x,s), \\ K^{+}(x,y;p,s) &= K^{-}(x,y;p,s), \end{aligned} \tag{13}$$

$$g^{+}(x,y;p,s) \equiv T^{+}_{A}(x,s)\ x + R^{+}_{A}(x,s)\ T^{-}_{B}(p,y)\ y, \tag{14}$$

$$g^{-}(x,y;p,s) \equiv T^{-}_{B}(p,y)\ y + R^{-}_{B}(p,y)\ T^{+}_{A}(x,s)\ x. \tag{15}$$

When one of the equations (12) is already solved, the solution of the other can be obtained directly by using corresponding explicit relation (11). Whereas the attention is drawn to the fact that in the equations (12):

1. The discussed explicit structures have until now met only in linear problems, with the ensuing advantages.
2. Furthermore, an increase of the intensity of external radiation that excites the medium, in the form of direct dependence appears only in free terms g^{+}, g^{-} of these equations. This, as well known, does not affect a convergence of the iterative solutions of considered equations, because it is due only to the properties of kernels.
3. The kernels of equations K^{+}, K^{-} are just the probability densities.

Aforesaid ensures a convergence, for example, of a simple iterative scheme

$$p^{(n+1)} = g^{+}_{(n)} + K^{+}_{(n)}\, p^{(n)} \quad \text{at} \quad s^{(0)} = y, \tag{16}$$

where

$$g^{+}_{(n)} \equiv g^{+}\left(x,\, y;\, p^{(n)},\, s^{(n)}\right), \quad K^{+}_{(n)} \equiv K^{+}\left(x,\, y;\, p^{(n)},\, s^{(n)}\right). \tag{17}$$

In the framework of the method of adding of layers, to determine the linear images of nonlinear PRT, the following sequential scheme can be distinguished: to begin with, we determine p and s from Eqs. (11)–(17), next it will be $p_{\pm}$ and $s_{\pm}$ from Eqs. (9)–(10), afterward $R^{\pm}_{L}(x,y)$, $T^{\pm}_{L}(x,y)$ from Eqs. (5)–(8), and finally $u^{\pm}_{L}(x,y)$ from Eqs. (3)–(4).

4 A complete set of equations of invariant imbedding for the linear images

As a second method for determining the linear images, we derive a complete set of equations of invariant imbedding. More consistent way is to fulfill a limiting transition in the general relations of addition of layers, which were built above, i.e. successively letting one layer be elementary $\Delta \to 0$, while the other is left fixed: $A \equiv \Delta$, $B \equiv L$ and $A \equiv L$, $B \equiv \Delta$. For radiation characteristics of diffuse reflection-transmission of elementary volume can be obtained the explicit forms

$$\begin{aligned} T^{\pm}_{\Delta}(x,y) &= 1 - æ^{\pm}(x,y)\,\Delta + O\left(\Delta^2\right), \\ R^{\pm}_{\Delta}(x,y) &= \chi^{\pm}(x,y)\,\Delta + O\left(\Delta^2\right). \end{aligned} \tag{18}$$

The physical meaning of the functions $æ^{\pm}(x,y)$ and $\chi^{\pm}(x,y)$ is as follows: they represent the probability densities that the quantum moving in a certain direction will first be absorbed by elementary layer of the medium, and then: a) $æ^{\pm}(x,y)$ will not be re-emitted in the same direction; b) $\chi^{\pm}(x,y)$ will be re-emitted in backwards. Hence a complete set of the equations of invariant imbedding can be written as follows:

$$\left[\frac{\partial}{\partial L} - \hat{E}_{+}\right] T^{+} = -T^{+}\, æ^{+}\left(x,u^{-}\right) + T^{+}\, \chi^{+}\left(x,u^{-}\right)\, R^{-}, \tag{19}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_{+}\right] R^{+} = T^{+}\, \chi^{+}\left(x,u^{-}\right)\, T^{-}, \tag{20}$$

$$\begin{aligned} \left[\frac{\partial}{\partial L} - \hat{E}_{+}\right] R^{-} = {} & \chi^{-}\left(x,u^{-}\right) - R^{-}\, æ^{+}\left(x,u^{-}\right) - \\ & - æ^{-}\left(x,u^{-}\right)\, R^{-} + R^{-}\, \chi^{+}\left(x,u^{-}\right)\, R^{-}, \end{aligned} \tag{21}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_{+}\right] T^{-} = -æ^{-}\left(x,u^{-}\right)\, T^{-} + R^{-}\, \chi^{+}\left(x,u^{-}\right)\, T^{-}, \tag{22}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_{-}\right] T^{+} = -æ^{+}(u^{+},y)\, T^{+} + R^{+}\, \chi^{-}(u^{+},y)\, T^{+}, \tag{23}$$

$$\begin{aligned} \left[\frac{\partial}{\partial L} - \hat{E}_{-}\right] R^{+} = {} & \chi^{+}(u^{+},y) - æ^{+}(u^{+},y)\, R^{+} - \\ & - R^{+}\, æ^{-}(u^{+},y) + R^{+}\, \chi^{-}(u^{+},y)\, R^{+}, \end{aligned} \tag{24}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_-\right] R^- = T^- \chi^-(u^+, y)\, T^+, \tag{25}$$

$$\left[\frac{\partial}{\partial L} - \hat{E}_-\right] T^- = -T^-\, æ^-(u^+, y) + T^-\, \chi^-(u^+, y)\, R^+. \tag{26}$$

The first quartet of equations is a consequence of variations of the left boundary of medium, and the second quartet is that of the right boundary. The corresponding operators of radiation "response" of medium can be written [5]

$$\hat{E}_+ = \alpha^+\left(x, u_L^-\right) \frac{\partial}{\partial x}, \quad \hat{E}_- = \alpha^-(u^+, y)\, \frac{\partial}{\partial y}, \tag{27}$$

where $\alpha^\pm$ are the well-known integral of collisions of the problem. Without going into details, we note that the initial conditions in the corresponding Cauchy problem, in terms of the parameter of layer thickness, are $R^\pm|_{L=0} = 0$, $T^\pm|_{L=0} = 1$, and in terms of the energy variables (x, y) – more particular solutions of PRT of single quantum, when the medium is excited by radiation incident only on one boundary (for details, see Example in Sect. 6).

5 Ambartsumian's functional equations for linear images

A third method of solution of PRT corresponds to the case when simultaneously vary both boundaries of the layer (when the elementary layer of infinitesimal thickness is added to one boundary, and it is subtracted from the other boundary). At this, a geometry of the problem is not changed, i.e. the layer thickness remained constant, so the derivatives of the spatial variables naturally should be excluded. By pairwise exclusion of derivatives over thickness from Eqs. (19)–(26), we obtain four functional equations of ACI for the linear images:

$$\begin{aligned} \hat{A}T^+ = {} & T^+\, æ^+\left(x, u^-\right) - æ^+(u^+, y)\, T^+ \\ & + R^+\, \chi^-(u^+, y)\, T^+ - T^+\, \chi^+\left(x, u^-\right)\, R^-, \end{aligned} \tag{28}$$

$$\begin{aligned} \hat{A}R^+ = {} & \chi^+(u^+, y) - æ^+(u^+, y)\, R^+ - R^+\, æ^-(u^+, y) \\ & + R^+\, \chi^-(u^+, y)\, R^+ - T^+\, \chi^+\left(x, u^-\right)\, T^-, \end{aligned} \tag{29}$$

$$\begin{aligned} \hat{A}R^- = {} & -\chi^-\left(x, u^-\right) + R^-\, æ^+\left(x, u^-\right) + æ^-\left(x, u^-\right)\, R^- - \\ & - R^-\, \chi^+\left(x, u^-\right)\, R^- + T^-\, \chi^-(u^+, y)\, T^+, \end{aligned} \tag{30}$$

$$\begin{aligned} \hat{A}T^- = {} & æ^-\left(x, u^-\right)\, T^- - T^-\, æ^-(u^+, y) \\ & + T^-\, \chi^-(u^+, y)\, R^+ - R^-\, \chi^+\left(x, u^-\right)\, T^-. \end{aligned} \tag{31}$$

The corresponding operator of radiation "response" of medium, when simultaneously vary both boundaries, i.e. the ACI operator, is given by $\hat{A} = \hat{E}_+ - \hat{E}_-$:

$$\hat{A} \equiv \alpha^+\left(x, u_L^-\right) \frac{\partial}{\partial x} - \alpha^-\left(u_L^+, y\right) \frac{\partial}{\partial y}. \tag{32}$$

It is noteworthy that the equations of linear images (20)–(31) favorably differed from the corresponding equations of previously known [5], $u_L^{\pm}(x,y)$, in the followings: (i) they retain a constructive explicit structure distinctive only for the equations of linear case, (ii) the characteristics of the elementary act of scattering (dependent on level of excitation of medium) are clearly separated from the structural forms, which are caused by the multiple scattering. The characteristics of the elementary act – $æ^{\pm}(x,y)$ and $\chi^{\pm}(x,y)$ at the transition to the linear case are converted into constant, when explicit structural forms, those just caused by the multiple scattering, are naturally retained. A transition to the functional equations of ACI (i.e. turn from the second method to the third, for determining the linear images) provides additional simplification. The layer thickness here are figured as fixed parameter for the whole calculation, whereas in the same problem with a given value of the layer thickness, the use of invariant imbedding necessarily implies an additional calculation of the entire family of PRT, starting from the value of zero thickness and continuing until reaching its final value, intended beforehand.

6 Particular example

Let us investigate next the simple instructive model of isotropic medium, with the conservative and isotropic scattering. Here we have the simplifications $R^{\pm} \equiv R$, $T^{\pm} \equiv T$, $R + T = 1$. The ACI equations, for determining the linear image T of function $u \equiv u(x,y)$, can be put in the simple symmetrical form

$$\left[k\,(x+v)\,\frac{\partial}{\partial x} + k\,(y+u)\,\frac{\partial}{\partial y}\right] T = -T\,M(x,y), \tag{33}$$

where

$$\begin{aligned} M(x,y) =& M(y,x) \equiv \frac{k\,(x+v) - k\,(y+u)}{x-y}, \\ k\,(\xi) \equiv& n\,\frac{h\nu}{2}\,\frac{A_{21}\,B_{12}}{A_{21} + \frac{\xi}{2}\,(B_{12}+B_{21})}, \end{aligned} \tag{34}$$

$$u = (x-y)\,T + y, \qquad v = -\,(x-y)\,T + x. \tag{35}$$

The initial conditions for Eq. (33) will be $T\,(x,0) = \sigma\,(x)$ or $T\,(0,y) = \sigma\,(y)$, where the unknown function $\sigma\,(z)$ describes the passage of a single quantum through medium, when it is excited by radiation of intensity z incident only on one boundary, and determined from its equation of invariant imbedding

$$\left[\frac{\partial}{\partial L} + x\,\sigma\,\frac{k\,(2\,x - x\,\sigma)}{2}\,\frac{\partial}{\partial x}\right]\sigma = -\sigma^2\,\frac{k\,(2\,x - x\,\sigma)}{2}, \tag{36}$$

$$\sigma|_{L=0} = 1, \quad \text{or} \quad \sigma|_{x=0} = q, \tag{37}$$

where q is the transmittance of layer of geometrical thickness L, in linear problem of isotropic medium at conservative isotropic scattering. It is explicitly given by

$$q = \frac{1}{1 + \frac{1}{2}\,k_0\,L}. \tag{38}$$

Thus, in this particular example, the following sequence of solutions of the problem we have in short: first solved a linear problem by means of (38), then this solution is used to define a linear image of a particular nonlinear PRT of unilateral illumination of medium by considering the auxiliary Cauchy problem (36)–(37) (by means of the equation of invariant imbedding), and afterward then the quasi-linear system of ACI (33)–(35) was considered. Hence, the desired solution of the nonlinear PRT, in term of its linear image of transmission of a single quantum, is given in an explicit form by (35).

7 Conclusion

In conclusion I want to express my deep appreciation to organizers of the conference in honor of bright memory and the 100th anniversary of academician V.V. Sobolev. For my great fascination by the theory of radiative transfer, I fully obliged to the two outstanding achievements of the field: the first is the "principle of invariance" of my teacher V.A. Ambartsumian, and the second is the "probabilistic interpretation" of V.V. Sobolev. Their incorporation provides the researchers by a powerful tools and methods of effective analysis of transfer problems, and a clear vision of their future opportunities. I am sure that the representatives of many more generations of astrophysicists, like me, would be fascinated by this area of knowledge.

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