Some New Directions of Development of the Radiative Transfer Theory

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It is shown that the problems of radiation transfer in homogeneous plane-parallel atmospheres admit a variational formulation, the equation of transfer then being the Euler–Lagrange equation and the known quadratic and bilinear relations being the conservation law due to form-invariance of the suitable Lagrangian. A group of transfer problems is revealed which are reducible to the source-free problem. We present a group-theoretical description of radiation transfer in inhomogeneous and multi-component atmospheres with plane-parallel geometry. The concept of composition groups is introduced for the media with different optical and physical properties. The group representations are derived for two possible cases of illumination of a composite finite atmosphere from outside. An algorithm for determining the global optical characteristics (reflectance and transmittance) of inhomogeneous and multi-component atmospheres is given. The group theory approach is also applied to determine the field of radiation inside the inhomogeneous atmosphere. The concept of a group of optical depth translations is introduced. The developed theory is illustrated with the problem of radiation diffusion with partial frequency distribution for the case where the inhomogeneity of the medium is due to the depth-variation of the scattering coefficient. It is shown that once reflectance and transmittance of a medium is determined, the internal field of radiation in the source-free atmosphere is found without solving any new equations.

1 Introduction

The research on the theory of radiative transfer carried out in recent two decades in Byurakan observatory develops Ambartsumian's ideas concerning the laws of addition of layers [1, 2] and the principle of invariance [2, 3, 4]. Being of importance for analytical theory itself, new results allow elaborating efficient computational schemes for various astrophysical applications involving radiation transfer in inhomogeneous absorbing and scattering atmospheres. In this context there is a need to define their place and importance in the modern transfer theory.

The report considers results obtained in two directions, the first of which concerns the variational formulation of radiation transfer problems in a plane-parallel homogeneous atmosphere.

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2 Lagrangian formalism

Before turning to immediate description of the variational or Lagrangian approach to radiative transfer problems we will briefly dwell on premises of this research. The fact is that although Ambartsumian's principle of invariance has been known for a long time, but its physical meaning remained obscure. In particular, it was unclear what are the limits of applicability and efficiency of the principle. The second point concerns Rybicki's work [5], where some quadratic integrals of the transfer equation were derived referred by him to as Q- and R-integrals. He supposed that these integrals are possibly related with the principle of invariance. In some problems they lead to non-linear relations linking to each other some characteristics of the radiation field in the atmosphere. Further generalization of Rybicki's results for monochromatic and isotropic scattering in a plane-parallel medium was given in [6, 7], where new sorts of relations were obtained referred to as bilinear and two-point bilinear relations, which couple the radiation fields at different depths of a given atmosphere as well as the radiation fields in different atmospheres.

In frameworks of variational formalism we developed the equations of transfer are proved to be none the other than the Euler-Lagrange equations and the non-linear Q-relations are the conservation laws due to form-invariance of the suitable Lagrangian. In fact, a single functional comprises all the information on features of the problem and allows a systematic connection between symmetries and conservation laws. Being the first integrals of the Euler–Lagrange equation, this laws may facilitate the solution of the problem under consideration and contribute to its interpretation.

To demonstrate the approach, we write the transfer equations in terms of the function Y having the following probabilistic meaning: it characterizes the probability of the photon exit from atmosphere in the direction μ, if originally it was moving at depth τ with the directional cosine η.

We have

$$\pm \frac{dY(\tau, \pm\eta, \mu)}{d\tau} = -Y(\tau, \pm\eta, \mu) + \frac{\lambda}{2} \int_{-1}^{1} Y(\tau, \pm\eta', \mu) d\eta', \tag{1}$$

where λ is the scattering coefficient. The Lagrangian density L corresponding to Eq. (1) was obtained in [8]

$$L(\Phi, \Phi', \tau, \eta, \mu) = \Phi^2 + (\eta\Phi')^2 - 2\Phi U, \tag{2}$$

where we introduced notations

$$\Phi(\tau, \eta, \mu) = Y(\tau, \eta, \mu) + Y(\tau, -\eta, \mu), \qquad U(\tau, \mu) = \frac{\lambda}{2} \int_{0}^{1} \Phi(\tau, \eta', \mu) d\eta'. \tag{3}$$

In accordance with the results of [8], the Euler–Lagrange equation has a form

$$\frac{\partial L}{\partial \Phi} - \frac{d}{d\tau} \frac{\partial L}{\partial \Phi'} + \lambda \int_{0}^{1} \frac{\partial L}{\partial U} d\eta' = 0. \tag{4}$$

One will make sure that insertion of the Lagrangian (2) into Eq. (4) yields the transfer equation (1). It is important that both the transfer equation (1) and the Lagrangian density (2) do not depend explicitly on τ, or stated differently, they are form-invariant under infinitesimal transformation of the optical depth.

This implies that the transformation (or translation) of the optical depth is the symmetry transformation for the system (1). The derivation of conservation laws from direct study of the variational integral is based on Noether's theorem (see, for instance, [9]), which was generalized in [10] to encompass the integro-differential equations. For the problem under consideration, it suggests a conservation law as follows:

$$\int_0^1 \left[L - \frac{\partial L}{\partial \Phi} \Phi' \right] d\eta = const, \tag{5}$$

which, in view of Eq. (2), takes a form

$$\int_0^1 Y(\tau,\zeta,\mu)\, Y(\tau,-\zeta,\mu) d\zeta = \frac{\lambda}{4} \left(\int_{-1}^1 Y(\tau,\zeta,\mu) d\zeta \right)^2 + const. \tag{6}$$

This relation is, in essence, a prototype of the Q-integral obtained by Rybicki [5]. The above considerations imply that by its content the integral (6) is an analog of the momentum conservation law in mechanics and is due to the axes translation transformation. It holds everywhere where λ does not vary with depth.

The variational formalism allows one not only to elucidate the physical meaning of invariance principle but enables to derive along with many known results a great number of new relations of great importance for the theory and applications. It allows one also to find out some statistical characteristics of the diffusion process in the atmosphere [7, 11]. Some of the known non-linear relations possess a fairly obvious physical or/and probabilistic meaning and can be written immediately on the base of simple arguments.

This approach reveals a group of common radiation transfer problems of astrophysical interest which admit quadratic and bilinear integrals. All of them can be reduced to the source-free problem. This group of problems referred to as RSF-problems includes Milne's problem, the problem of diffuse reflection (and transmission in the case of the atmosphere of finite optical thickness) as well as problems with exponential and polynomial laws for the distribution of internal energy sources. The group problems are characterized at least by three features. First of all, the invariance principle implies bilinear relations connecting the solutions of the listed problems. It was shown in [12] that the group of the RSF-problems admits a class of integrals involving quadratic and bilinear moments of the intensity of arbitrarily high orders. Secondly, if the problem can be formulated for finite atmosphere then the principle allows connecting its solution with that of the proper problem for a semi-infinite atmosphere. Finally, knowledge of the

Ambartsumian φ-function reduces their solutions to the Volterra-type equations for the source function with the kernel-function

$$L(\tau) = \frac{\lambda}{2} \int_0^1 \varphi(\zeta) e^{-\frac{\tau}{\zeta}} \frac{d\varphi}{\zeta}. \tag{7}$$

While the variational approach is widely used in various branches of theoretical physics, it was not the case in the field of the radiative transfer theory, with the only exception being the paper of Anderson [8] who established the conservation law suitable for the case of non-isotropic scattering. We used the results of the rigorous mathematical theory in applying the Lagrangian formalism to the one-dimensional transfer problem [13].

3 Group-theoretical description of radiative transfer in inhomogeneous atmospheres

The next topic of the report concerns application of the group theory to solve the radiative transfer problems in inhomogeneous atmospheres under general assumptions on the frequency-angle distribution of the radiation field, the elementary event of scattering and properties of the medium. As we shall see, the theory we put forward can be regarded as a further extension of the layers adding method proposed first by Ambartsumian [1, 2] for one-dimensional homogeneous media and generalized by Nikoghossian [14, 15] over the case of inhomogeneous media. We remind that the method establishes summation laws for global optical properties of absorbing and scattering media (reflectance and transmittance), which express these properties of the combined medium through similar properties of its components. Of special interest is the particular limiting case of this method when optical thickness of one of the added components tends to zero. This allows one to find the global optical characteristics of a medium simultaneously for a family of the media of different thicknesses. This branch of the theory was developed by Bellman and his co-authors (see, e.g., [16, 17]) and is known as "invariant imbedding".

3.1 Composition groups

We start with considering the amalgamation procedure of the plane-parallel absorbing and scattering inhomogeneous media. It is assumed that the added components do not contain primary energy sources and are allowed to differ one from the other not only by optical thicknesses, but also by the nature of inhomogeneity. By inhomogeneity we mean that each of the physical parameters specifying the elementary event of scattering or physical state of the medium may vary with depth. Of them we note the profile of the absorption coefficient, the quantum scattering (or destruction) coefficient, Voigt's parameter, the phase function, the frequency redistribution function, the Stokes parameters in the case of polarized radiation, the correlation length for turbulent media, and so on. However, in illustrating the approach, we restrict ourselves by treating the 1D transfer problem for the case of partial redistribution over frequencies by assuming that the only variable parameter is the scattering coefficient.

Now we introduce the concept of composition or transformation of scattering and absorbing inhomogeneous media, which refers to the addition of a new medium to the initial one. The transformations induced in this way form a group if under the group product (binary operation) one takes the resultant of two successive transformations. It is remarkable that this definition does not specify the nature of inhomogeneity of added media. It is easily seen that all the required conditions for forming a group are satisfied. In particular, the role of the unit element is played by the identity transformation, which leaves the initial medium unchanged, and the inverse element is the transformation which reverses the effect of the already performed transformation. The associativity of the group product is obvious. We refer to this group of transformations as the GN(2,C) group, which, evidently, is not commutative. As a result of the described compositions, one can construct different atmospheres composed of inhomogeneous components.

Of special interest is one of subgroups of the introduced group which describes the case when the added media are homogeneous. The components of such a composite atmosphere may differ from each other not only by optical thicknesses but also by any characteristics of the radiation diffusion in them. Such groups, referred nominally to as GNH(2,C), are two-, three- and multi-parameter dependent on the number of parameters changing in passing from one component to another. The groups of these types are infinite and non-commutative. They can serve as archetypes for a number of real radiating media of astrophysical importance. Finally, of independent interest is the narrower subgroup of the introduced two groups which involves compositions of homogeneous media with identical physical properties but, in general, of different optical thicknesses. These compositions obviously yield homogeneous medium. This one-parameter group, we call it GH(2,C), is infinite and commutative, i.e., Abelian [18]. It becomes continuous when the only parameter, optical thickness, varies continuously.

3.2 The group representations

In order to find the representations of introduced groups, consider a composite atmosphere consisted of two layers, which generally differ in both the optical thickness and functional behavior of parameters specifying the elementary event of scattering (Fig. 1). This means that both components are inhomogeneous and possess the property of polarity [14]. The scattering in the media is supposed occurring with redistribution over directions and frequencies so that the optical characteristics of media may be presented in the operator-matrix form with the matrix elements possessing probabilistic meaning (throughout the paper we use the probability language). They describe the angle and/or frequency dependent probabilities of a single event of reflection and transmission. Having in mind

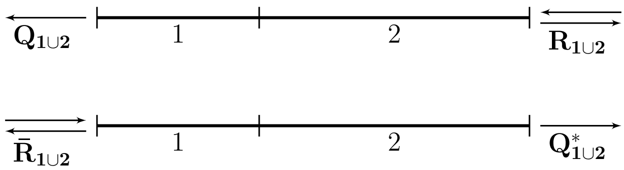

Figure 1: Reflection and transmission by inhomogeneous atmosphere.

the polarity property of inhomogeneous media, we introduce the notations $\mathbf{R}_i$, $\mathbf{Q}_i$ and $\bar{\mathbf{R}}_i$, $\bar{\mathbf{Q}}_i$ $(i = 1, 2)$ for the reflection and transmission coefficients of the components of a composite medium illuminated correspondingly from the right and left. In accordance with the principle of reversibility of optical phenomena, $\bar{\mathbf{Q}}_i = \mathbf{Q}_i^*$, where the transposed matrix is supplied by asterisk. Everywhere below we follow the designation $\mathbf{Q}_i^*$. An important role in this research belongs to the inverse of the transmittance matrix $\mathbf{P} = \mathbf{Q}^{-1}$ and the other three combined matrices $\mathbf{S} = \mathbf{RP}$, $\bar{\mathbf{S}} = \mathbf{P}\bar{\mathbf{R}}$, $\mathbf{M} = \mathbf{Q}^* - \mathbf{S}\bar{\mathbf{R}}$. These four matrices provide a complete description of the optical properties of an inhomogeneous absorbing and scattering medium independent of that what of its boundaries is illuminated from outside.

Let us treat now the transfer of radiation through composite medium when a photon falls on its right boundary (top drawing in Fig. 1). Taking account of possibility of multiple reflections between components of the medium, one can derive the following two relations (see [19]):

$$\mathbf{P}_{1\cup 2} = \mathbf{P}_2\mathbf{P}_1 - \bar{\mathbf{S}}_2\mathbf{S}_1, \tag{8}$$

$$\mathbf{S}_{1\cup 2} = \mathbf{S}_2\mathbf{P}_1 + \mathbf{M}_2\mathbf{S}_1, \tag{9}$$

where the quantities pertaining to composite medium are indexing with $1 \cup 2$.

Taking together, relations (8) and (9) can be presented in the more convenient compact form

$$\begin{pmatrix} \mathbf{P}_{1\cup 2} \\ \mathbf{S}_{1\cup 2} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_2 & -\bar{\mathbf{S}}_2 \\ \mathbf{S}_2 & \mathbf{M}_2 \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{S}_1 \end{pmatrix}, \tag{10}$$

where we used the concepts of supervector and supermatrix [18, 20, 21]. The supermatrix entering in Eq. (10) is denoted by $\tilde{\mathbf{A}}$ (hereafter the supermatrices are supplied by tilde)

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{P} & -\bar{\mathbf{S}} \\ \mathbf{S} & \mathbf{M} \end{pmatrix}. \tag{11}$$

The set of matrices $\tilde{\mathbf{A}}$ is the first of representations of the group of compositions GN(2,C) which also is a group (we denote it by g) and provides a one-to-one mapping of GN(2,C) to supervector space, i.e., the group product of two transformations $g_1 \bigotimes g_2$, corresponds to $\tilde{\mathbf{A}}_{1\cup 2} = \tilde{\mathbf{A}}_1\tilde{\mathbf{A}}_2$, or for representations $\Im(g_1 \bigotimes g_2) = \Im(g_1)\Im(g_2)$ (isomorphism). On the hand, the supermatrix $\tilde{\mathbf{A}}$ can be regarded as an operator mapping one supervector space to another one.

It is natural to refer nominally to this supermatrix as "composer". It plays an important role in the developed theory.

It is easy to see that the transformation realizing by $\tilde{\mathbf{A}}$ provides determination of optical properties of the composed medium partially, namely, only those for the right-hand side illumination. For complete description of optical properties of the composite medium, we need the matrices $\bar{\mathbf{S}}$ and $\mathbf{M}$ which obey the following transformations [19]:

$$\bar{\mathbf{S}}_{1\cup 2} = \mathbf{P}_2\bar{\mathbf{S}}_1 + \bar{\mathbf{S}}_2\mathbf{M}_1, \qquad \mathbf{M}_{1\cup 2} = \mathbf{M}_2\mathbf{M}_1 - \mathbf{S}_2\bar{\mathbf{S}}_1. \tag{12}$$

Note that these relations could be derived directly.

In the matrix-operator form they read

$$\begin{pmatrix} \mathbf{M}_{1\cup 2} \\ \mathbf{S}_{1\cup 2} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_2 & -\mathbf{S}_2 \\ \bar{\mathbf{S}}_2 & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} \mathbf{M}_1 \\ \bar{\mathbf{S}}_1 \end{pmatrix}. \tag{13}$$

Thus, we are led to an alternative group of representations given by the supermatrix

$$\tilde{\mathbf{B}} = \begin{pmatrix} \mathbf{M} & -\mathbf{S} \\ \bar{\mathbf{S}} & \mathbf{P} \end{pmatrix}, \tag{14}$$

which we denote by $\bar{\Im}(g)$. It is evident that this group also is isomorphic to the group of compositions GN(2,C) and together with $\Im(g)$ gives a complete description of optical properties of the composite atmosphere illuminated from the right. In both cases the identity transformation is given by the supermatrix

$$\tilde{\mathbf{E}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{15}$$

where $\mathbf{I}$ is the unit matrix. The supermatrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ are non-degenerate, and two-sided inverse matrices exist with superdeterminant [21, 22, 23] equaled to one (see [19]).

By introducing the four-dimensional supervector $\tilde{\mathbf{Y}}$ with the components $(\mathbf{P}, \mathbf{S}, \mathbf{M}, \bar{\mathbf{S}})$, the group representations $\Im(g)$, $\bar{\Im}(g)$ can be joined and presented as a reducible representation

$$\tilde{\mathbf{Y}}_{1\cup 2} = \tilde{\mathbf{\Psi}}_2\tilde{\mathbf{Y}}_1, \tag{16}$$

where

$$\tilde{\mathbf{\Psi}} = \begin{pmatrix} \mathbf{P} & -\bar{\mathbf{S}} & \mathbf{0} & \mathbf{0} \\ \mathbf{S} & \mathrm{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{M} & -\mathbf{S} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{S}} & \mathbf{P} \end{pmatrix}. \tag{17}$$

We conclude that, given the optical properties of the component layers, the common matrix multiplications allow one to determine these properties for the compound atmosphere. If the atmosphere is homogeneous one can restrict oneself by transformation Eq. (10). Arguments analogous to those above in deriving Eq. (17) allow one to derive adding laws for the case when the composite atmosphere is illuminated from the side of the left boundary (bottom drawing in Fig. 1) [19].

3.3 The 1D source-free problem for partial redistribution over frequencies

Consider a subgroup of the composition group GNH(2,C) subjected to the only condition that the optical thickness of the medium obtained as a result of compositions must not exceed some presetting value of τ_0. When the optical thickness varies continuously, this infinite group is obviously continuous. Then this group together with its representation $\Im(g)$ are one-dimensional Lie groups [21, 22, 23]. With help of compositions of this groups one can construct a multi-component atmosphere with components which generally can differ one from the other by their physical characteristics.

As an example, let us treat the matrix problem of radiation diffusion in a one-dimensional inhomogeneous atmosphere illuminated from the boundary $\tau = \tau_0$ when the scattering obeys the angle averaged law of partial redistribution over frequencies. Suppose that the atmosphere consists of components of equal and sufficiently small thickness characterized by some constant values of the scattering coefficient λ, so that in the limit of the components thicknesses tending to zero it might be regarded as a continuous function of the optical depth.

The infinitesimal operator of this group of compositions at τ_0 can be represented in the form

$$\tilde{\boldsymbol{\Xi}}(\tau_0) = \lim_{\Delta\tau_0 \to 0} \frac{\widetilde{\mathbf{A}}(\tau_0 + \Delta\tau_0) - \widetilde{\mathbf{A}}(\tau_0)}{\Delta\tau_0} = \begin{pmatrix} \mathbf{m}(\tau_0) & -\mathbf{n}(\tau_0) \\ \mathbf{n}(\tau_0) & -\mathbf{m}(\tau_0) \end{pmatrix}, \tag{18}$$

where

$$\mathbf{m}(\tau_0) = \alpha - \mathbf{n}(\tau_0), \qquad \mathbf{n}(\tau_0) = \frac{\lambda(\tau_0)}{2}\boldsymbol{\Gamma}. \tag{19}$$

Here α and $\boldsymbol{\Gamma}$ are the discrete analogs correspondingly of the profile of the absorption coefficient and the law of the frequency redistribution [24]. For the sake of simplicity, they are supposed to be independent of depth. Evidently, $\boldsymbol{\Gamma}$ is a symmetric matrix and α is a diagonal matrix with the elements $\alpha_i = \alpha(x_i)$.

Transformation (8) implies [25]

$$\frac{d\mathbf{P}}{d\tau_0} = \mathbf{m}(\tau_0)\,\mathbf{P}(\tau_0) - \mathbf{n}(\tau_0)\,\mathbf{S}(\tau_0), \tag{20}$$

$$\frac{d\mathbf{S}}{d\tau_0} = \mathbf{n}(\tau_0)\,\mathbf{P}(\tau_0) - \mathbf{m}(\tau_0)\,\mathbf{S}(\tau_0), \tag{21}$$

with the initial conditions $\mathbf{P}(0) = \mathbf{I}$, $\mathbf{S}(0) = \mathbf{0}$, where $\mathbf{0}$ is the null matrix.

Inversion of the matrix $\mathbf{P}(\tau_0)$ found from the set of equations (20) and (21) allows one to determine the requisite values of the medium reflectance and transmittance. Analogously, by using the infinitesimal operator of the supermatrix $\tilde{\mathbf{B}}$ and Eq. (14), we are led to a new set of the matrix differential equations

$$\frac{d\mathbf{M}}{d\tau_0} = -\mathbf{m}(\tau_0)\,\mathbf{M}(\tau_0) - \mathbf{n}(\tau_0)\,\bar{\mathbf{S}}(\tau_0), \tag{22}$$

$$\frac{d\bar{\mathbf{S}}}{d\tau_0} = \mathbf{n}(\tau_0)\,\mathbf{M}(\tau_0) + \mathbf{m}(\tau_0)\,\bar{\mathbf{S}}(\tau_0), \tag{23}$$

with the initial conditions $\mathbf{M}(0) = \mathbf{I}$, $\bar{\mathbf{S}}(0) = \mathbf{0}$.

In the case of homogeneous atmosphere one can restrict oneself to solving the set of equations (20)–(21). Its solution can be presented in the form of the matrix exponential [25]. Note that from the sets of equations (20)–(23) one can derive separate matrix differential equations of the second order for unknown matrix-functions as it is the case in the scalar case [25].

Equations obtained with the group approach exhibit intimate connection between the group approach and the method of invariant imbedding [16, 17]. As a matter of fact, the invariant imbedding technique is equivalent to action of infinitesimal operators of the proper group representations introduced in the paper. For homogeneous atmosphere, the obtained equations admit invariants or conservation laws, the continual analogs of which were obtained in the mentioned papers [7, 8, 25, 26].

The efficiency of the developed theory becomes especially discernible when solving radiative transfer problems for atmospheres with a complex multi-layer structure. In applying any of the introduced composers, one needs to predetermine the global optical properties of each of the layers added to the boundary $\tau = \tau_0$, namely, the matrices $\mathbf{P}$, $\mathbf{S} = \mathbf{RP}$, $\bar{\mathbf{S}} = \mathbf{P}\bar{\mathbf{R}}$ and $\mathbf{M} = \mathbf{Q}^* - \mathbf{S}\bar{\mathbf{R}} = \mathbf{Q}^* - \mathbf{R}\bar{\mathbf{S}}$, i.e., the triad of matrices $\mathbf{R}$, $\bar{\mathbf{R}}$, $\mathbf{Q}$. The problem is simpler when the components are homogeneous. Particularly, in the scalar problems these quantities are determined analytically. In the general case of inhomogeneous components, we can turn to solutions of the systems of equations (20)–(23) with subsequent inversion of the matrix $\mathbf{P}$. This route is preferable in finding the field of radiation inside the medium to be discussed below. However, there exists an alternative way of determining the required optical properties by solving basic differential equations obtained in [12, 27], which are easily realizable initial-value problems.

Thus, the algorithm of solution of the transfer problem in the most general case of multi-component atmosphere is as follows. One starts with finding the reflectance and transmittance of the layers to be added by using one of the routes described above. Further, the compositions transformations are continued until the optical thickness of the composite atmosphere specified by the problem formulation is attained. Inversion of the matrix $\mathbf{P}(\tau_0)$ allows one to find $\mathbf{Q}(\tau_0)$ what, in its turn, determines other properties of the composite atmosphere. We shall see below that the obtained quantities are sufficient to find the field of radiation inside the medium.

In the special case when the supplemented layers are homogeneous and possess similar properties, we deal with the cyclic group and the composition process reduces to the action of powers of corresponding operators ($\tilde{\mathbf{A}}^n$, for instance). This naturally reduces the volume of computations to a great extent.

3.4 Radiation field inside the medium

The goal we pursue in this section is to extend the group theory approach over the field of radiation inside inhomogeneous media. Consider a plane-parallel inhomogeneous atmosphere of optical thickness τ_0, the boundary $\tau = \tau_0$ of which is illuminated from outside (Fig. 2). Light scattering is generally assumed occurring with the angle and frequency redistribution. The internal field of radiation we assign by the matrices $\mathbf{U}(\tau, \tau_0)$ and $\mathbf{V}(\tau, \tau_0)$, which specify the probabilities that the quantum with the angle-frequency characteristics (η, x) falling on the boundary $\tau = \tau_0$, will be found, as a result of diffusion in the medium, at the depth τ moving correspondingly to the boundaries $\tau = 0$, and $\tau = \tau_0$, generally with some other characteristics (η', x').

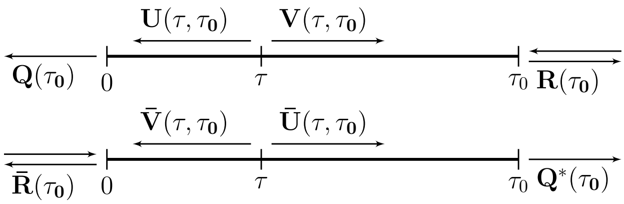

Figure 2: Description of the radiation field inside the inhomogeneous atmosphere.

Let us treat now the procedure of transition from one optical depth to another one by supplementing a new layer. The infinite set of such transitions obviously composes a group if the group product is defined as the result of two subsequent transitions. One can easily check that all the group postulates are satisfied. In accordance with the physics of the problem, the resulting value of the optical depth should not exceed the optical thickness of the medium $\tau \leq \tau_0$. This group is a subgroup of the group GN(2,C) and is equivalent to the similar subgroup considered in the preceding sections for composition of different media.

Taking into account the probability meaning of matrices $\mathbf{U}(\tau, \tau_0)$ and $\mathbf{V}(\tau, \tau_0)$, one can write

$$\mathbf{Q}(\tau_0) = \mathbf{Q}(\tau)\,\mathbf{U}(\tau, \tau_0), \quad \mathbf{V}(\tau, \tau_0) = \mathbf{R}(\tau)\,\mathbf{U}(\tau, \tau_0), \tag{24}$$

hence

$$\mathbf{U}(\tau, \tau_0) = \mathbf{P}(\tau)\,\mathbf{Q}(\tau_0), \quad \mathbf{V}(\tau, \tau_0) = \mathbf{S}(\tau)\,\mathbf{Q}(\tau_0). \tag{25}$$

The fact of separation of arguments in $\mathbf{U}(\tau, \tau_0)$ and $\mathbf{V}(\tau, \tau_0)$ is one of advantages of the applied approach. Equations (24) imply that the subgroup of representation $\Im(g)$ relevant to the media compositions group may be now regarded as representation of the depth-translation group.

Indeed, on the base of Eq. (10), one may write

$$\begin{pmatrix} \mathbf{U}(\tau + \delta\tau, \tau_0) \\ \mathbf{V}(\tau + \delta\tau, \tau_0) \end{pmatrix} = \begin{pmatrix} \mathbf{P}_\tau(\delta\tau) & -\bar{\mathbf{S}}_\tau(\delta\tau) \\ \mathbf{S}_\tau(\delta\tau) & \mathbf{M}_\tau(\delta\tau) \end{pmatrix} \begin{pmatrix} \mathbf{U}(\tau, \tau_0) \\ \mathbf{V}(\tau, \tau_0) \end{pmatrix}, \tag{26}$$

where $\delta\tau$ is an increment to the optical depth τ. The subscript τ indicates that the internal physical properties of supplemented layer are relevant to (or vary in) the interval $(\tau, \tau + \delta\tau)$.

Thus, the supermatrix $\tilde{\mathbf{A}}$ plays an important role not only in adding the media of different optical thicknesses but also in translating optical depths inside inhomogeneous atmosphere. Stating differently, it serves at the same time as "composer" of inhomogeneous atmospheres and as "translator" in transitions between optical depths inside the atmosphere. It is noteworthy that in the latter case only the global optical properties of the incremented layer provide the transformations. The internal physical characteristics do not take an immediate part in these transformations, so that the nature of inhomogeneity in different media or layers are allowed to be different.

To illustrate the obtained results, let us return to the matrix case of the transfer problem treated in Section 3.3, where we confined ourselves to the global optical characteristics of the medium. Our immediate objective now is to find the field of radiation inside the medium, where, again, the only parameter varying with depth is the scattering coefficient λ. In light of that said in Sect. 3.3, we conclude that the depth-translation group together with its representation are the Lie groups of the one-dimension.

Given the supermatrix (18), the transformation (26) leads to the customary differential equations of radiation transfer for the operator-functions $\mathbf{U}$ and $\mathbf{V}$

$$\frac{d\mathbf{U}}{d\tau_0} = \mathbf{m}(\tau)\,\mathbf{U}(\tau,\tau_0) - \mathbf{n}(\tau)\,\mathbf{V}(\tau,\tau_0), \tag{27}$$

$$\frac{d\mathbf{V}}{d\tau_0} = \mathbf{n}(\tau)\,\mathbf{U}(\tau,\tau_0) - \mathbf{m}(\tau)\,\mathbf{V}(\tau,\tau_0). \tag{28}$$

In place of the usual boundary conditions, one can now adopt the conditions at $\tau = \tau_0$, $\mathbf{U}(0,\tau_0) = \mathbf{Q}(\tau_0)$, $\mathbf{V}(0,\tau_0) = \mathbf{0}$, then reducing the problem to that with initial conditions. Derivation of the transfer equations (27)–(28) on the base of physical reasoning is straightforward, what is usually doing in the classical astrophysical literature. As it was shown, the operator-functions $\mathbf{P}(\tau)$ and $\mathbf{S}(\tau)$ satisfy the same set of equations (20)–(21) with the initial conditions $\mathbf{P}(0) = \mathbf{I}$, $\mathbf{S}(0) = \mathbf{0}$. By comparing the initial conditions of these two sets of equations, we are led to relations (24) written above on the base of probabilistic reasoning [26].

Bearing in mind the computations described in Section 3.4 for the composite inhomogeneous atmosphere as well as the equivalence of the medium-composition and the depth-translation subgroups of GNH(2,C), we arrive at an important conclusion that the internal field of radiation now can be found without solving any new equations. Indeed, it is sufficient to this end to multiply the obtained value of $\mathbf{Q}(\tau_0)$ by $\mathbf{P}$ and $\mathbf{S}$ found above in intermediate calculations in constructing the atmosphere under study.

The far reaching analogy between media composition and depths translation groups makes it possible to transfer different results obtained for global optical properties of an atmosphere to quantities determining the internal field of radiation. For instance, if the atmosphere is homogeneous, one can derive conservation laws in terms of **U** and **V**, as it was done above for the matrices **P** and **S**. We do not deal with it here but refer the interested reader after continual analogs of these laws to [7, 26].

4 Conclusions

We discussed two directions of further development of the radiation transfer theory which, in our opinion, are promising from both the analytical and computational points of view. They generalize Ambartsumian's ideas concerning the principle of invariance and the layers adding laws. The variational approach allows one to reveal the physical nature and the scope of applicability of invariance principle. It is important that the solutions of some standard problems of astrophysical interest mathematically are reducible to the Volterra type integral equations.

The second direction concerns the group theory which is applied to solve the problems of radiative transfer in inhomogeneous absorbing and scattering atmospheres. The media composition groups and their representations introduced in the paper generalize the layers adding approach, which now covers inhomogeneous, particularly multi-component, atmospheres with allowance of the angle and frequency distribution of the radiation field. The group representations being expressed in terms of some combined discrete quantities allow one to find the most general summation laws for reflectance and transmittance of the plane-parallel media.

Employment of infinitesimal operators of the introduced groups makes it possible to establish the close connection of the introduced groups with the classical transfer equations and the equations ensuing from invariant imbedding. In fact, the first of them are connected with the depth translation groups, while the second – with composition groups for the media of different optical thicknesses.

An important result in considering the internal field of radiation is the separation of variables of the optical depth and thickness in the expression of quantities describing the optical properties. This implies that the introduced group of the optical depths translations is a subgroup of the group of the media compositions. In its turn, this means that after finding the reflectance and transmittance of an atmosphere, there is no need to solve any new equations to determine the internal field of radiation in the source-free atmosphere.

The theory we put forward is of sufficiently great generality since it does not depend on the nature of inhomogeneity of the media as well as on the angle and frequency distribution of the radiation field.

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