About the Development of the Asymptotic Theory of Non-Stationary Radiative Transfer

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A brief review of the development of the asymptotic non-stationary radiative transfer theory is presented. In particular, the accuracy of the diffusion approximation is studied. It is shown that the replacement of the non-stationary transfer equation by the heat conductive equation should give satisfactory results when the single scattering albedo λ is close to the unity. But this approximation can lead to significant errors when $\lambda < 1$.

Studying time-dependent processes in various non-stationary objects is an important problem of modern astrophysics. The illumination of the dust nebula under the influence of radiation of a new star can be considered as an example of such process.

Sobolev initiated the systematic development of the theory of non-stationary radiation fields in the article [1] published in 1952. Fundamentals of this theory were presented in his book [2].

Non-stationary radiation fields are characterized by the finite speed of light c and a definite duration of the light scattering process.

Let t_1 be the mean time of the stay of a photon in the absorbed state. It is usually assumed that the probability of emission of a photon being in the absorbed state in the time interval from t to $t + dt$ depends on t by the exponential law, i.e., it is proportional to $e^{-\frac{t}{t_1}} \frac{dt}{t_1}$.

The probability of the photon absorption while travelling after his radiation during an interval of time from t to $t + dt$ depends on t also exponentially, e.g., it is proportional to $e^{-\frac{t}{t_2}} \frac{dt}{t_2}$, where $t_2 = \frac{1}{\alpha c}$ is the mean time of stay of a photon on the path between two consecutive scatterings. Here α is the volume absorption coefficient of the medium.

The values of t_1 and t_2 are usually very different from each other. Therefore, Sobolev has proposed to allocate the consideration of two limiting cases, i.e., the case A, when $t_1 \gg t_2$, and the case B, when $t_2 \gg t_1$.

The simplest model of non-stationary radiative transfer is a model based on the consideration of the one-dimensional homogeneous infinite medium with an energy source depending on time. Let us assume that the medium is illuminated by a momentary point source of luminosity L flashing at some initial moment of time. We note that an actual flash duration and a dependence of the luminosity $L(t)$

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on the time t can be taken into account by integrating over time the expressions for light field characteristics found in the case of a point source multiplied by the value of the luminosity $L(t)$.

Let $I_1(r,t)$ and $I_2(r,t)$ be intensities of the radiation spreading on distance r from the source at time t in the direction of increasing and decreasing values of the coordinate r, respectively. Instead of the geometric distances r, the physical time t and values t_1 and t_2, we use the corresponding dimensionless quantities

$$\tau = \alpha r, \quad u = \frac{t}{t_1 + t_2}, \quad \beta_1 = \frac{t_1}{t_1 + t_2}, \quad \beta_2 = \frac{t_2}{t_1 + t_2}. \tag{1}$$

Then the radiative transfer equation takes the following form:

$$\frac{\partial I_1(\tau,u)}{\partial \tau} + \beta_2 \frac{\partial I_1(\tau,u)}{\partial u} = -I_1(\tau,u) + B(\tau,u), \tag{2}$$

$$-\frac{\partial I_2(\tau,u)}{\partial \tau} + \beta_2 \frac{\partial I_2(\tau,u)}{\partial u} = -I_2(\tau,u) + B(\tau,u). \tag{3}$$

Here $B(\tau,u)$ is the source function defined by the equation of radiative equilibrium

$$B(\tau,u) = \frac{\lambda}{2} \int_0^u \left[I_1\left(\tau,u'\right) + I_2(\tau,u')\right] e^{-\frac{u-u'}{\beta_1}} \frac{du'}{\beta_1}, \tag{4}$$

where λ is the single scattering albedo. These equations are supplemented with the initial condition which takes into account the momentary point source of energy. The mean radiation intensity $J(\tau,u)$ and the radiation flux $H(\tau,u)$ are defined by the expressions

$$J(\tau,u) = \frac{1}{2}\left[I_1(\tau,u) + I_2(\tau,u)\right], \tag{5}$$

$$H(\tau,u) = I_1(\tau,u) - I_2(\tau,u). \tag{6}$$

Minin [3] obtained the exact solution of this problem by means of the Laplace transform.

Simple asymptotic expressions for characteristics of the non-stationary radiation field are obtained in the case when points of the medium are located at large optical distances from energy sources ($\tau \gg 1$) and scattering of light is close to conservative ($1-\lambda \ll 1$). In this case Minin [4] proposed to use a simple technique for inverting the Laplace transform. As it is known from the theory of the Laplace transform, the value of the original at large values of the argument ($u \gg 1$) is determined using the expansion of the image in powers of the small parameter s. This expansion corresponds to the expansion of solutions of the stationary radiative transfer equation in powers of the small values of $1-\lambda$. As a result of the Laplace transform in time, the non-stationary equation is converted into the stationary one but the value of λ is replaced by the value $\frac{\lambda}{(1+\beta_1 s)(1+\beta_2 s)}$. Therefore, taking into account the fact that $\beta_1 + \beta_2 = 1$, we obtain $1 - \lambda = s$ with accuracy to members of the higher degrees of the parameter s. Hence, when receiving the asymptotic image, it is necessary to replace the small values of $1 - \lambda$ by s in the equation solution for the stationary case, and then to apply the inverse Laplace transform.

In the case of one-dimensional infinite medium illuminated by a momentary point source, we obtain for $J(\tau, u, \lambda)$ and $H(\tau, u, \lambda)$ the following expressions (for $\lambda = 1$, $\tau \gg 1$, $u > \tau$):

$$J_D\,(\tau, u, 1) = \frac{L}{4\sqrt{\pi u}}\; e^{-\frac{\tau^2}{4u}}, \tag{7}$$

$$H_D\,(\tau, u, 1) = \frac{L}{4\sqrt{\pi u}}\;\frac{\tau}{u}\; e^{-\frac{\tau^2}{4u}}. \tag{8}$$

The same expressions for these quantities are obtained in the diffusion approximation in the case of $\lambda = 1$. This approximation is based on using the heat conductivity equation

$$\frac{\partial^2 J(\tau, u, \lambda)}{\partial \tau^2} = \frac{\partial J(\tau, u, \lambda)}{\partial u} + (1 - \lambda)\, J(\tau, u, \lambda) \tag{9}$$

instead of the non-stationary radiation transfer equation. The solution of the equation (9) leads to the following expressions for the functions $J(\tau, u, \lambda)$ and $H(\tau, u, \lambda)$:

$$J_D\,(\tau, u, \lambda) = e^{-(1-\lambda)u} J_D\,(\tau, u, 1)\,, \tag{10}$$

$$H_D\,(\tau, u, \lambda) = e^{-(1-\lambda)u} H_D\,(\tau, u, 1)\,, \tag{11}$$

where $J_D\,(\tau, u, 1)$ and $H_D\,(\tau, u, 1)$ are given by the expressions (7) and (8).

The diffusion approximation was proposed by Compton [5] in 1923. However, in 1926 Milne [6] showed that the usage of this approximation for the calculation of non-stationary fields of radiation can lead to physically unreasonable results.

Kolesov and Sobolev [7] studied the accuracy of the diffusion approximation in the cases A and B.

Exact expressions for $J(\tau, u, \lambda)$ and $H(\tau, u, \lambda)$ in the case A have the form

$$J_A\,(\tau, u, \lambda) = \frac{L}{2\pi} \int_0^\infty e^{-\left(1-\frac{\lambda}{1+x^2}\right)u}\; \frac{\cos x\tau}{1+x^2}\, dx, \tag{12}$$

$$H_A\,(\tau, u, \lambda) = \frac{L}{\pi} \int_0^\infty e^{-\left(1-\frac{\lambda}{1+x^2}\right)u}\; \frac{x \sin x\tau}{1+x^2}\, dx. \tag{13}$$

When $\lambda u \gg 1$, we have the following asymptotic expressions:

$$J_A^{as}(\tau, u, \lambda) \approx \frac{L}{4\sqrt{\pi \lambda u}}\, e^{-(1-\lambda)u - \frac{\tau^2}{4\lambda u}}, \tag{14}$$

$$H_A^{as}(\tau, u, \lambda) \approx \frac{L}{4\sqrt{\pi \lambda u}}\, \frac{\tau}{\lambda u} e^{-(1-\lambda)u - \frac{\tau^2}{4\lambda u}}. \tag{15}$$

In the absence of true absorption, when $\lambda = 1$, these expressions coincide with the expressions (7) and (8) of the diffusion approximation.

In the case B for $\tau \geq 0$ and $u > \tau$, the exact expressions for these quantities are given by the expressions

$$J_B(\tau, u, \lambda) = \frac{\lambda L}{8}\left[I_0\left(\frac{\lambda}{2}\sqrt{u^2 - \tau^2}\right) + \frac{u}{\sqrt{u^2 - \tau^2}} I_1\left(\frac{\lambda}{2}\sqrt{u^2 - \tau^2}\right)\right] e^{-\left(1 - \frac{\lambda}{2}\right)u}, \tag{16}$$

$$H_B(\tau, u, \lambda) = \frac{\lambda L}{4}\frac{\tau}{\sqrt{u^2 - \tau^2}} I_1\left(\frac{\lambda}{2}\sqrt{u^2 - \tau^2}\right) e^{-\left(1 - \frac{\lambda}{2}\right)u}, \tag{17}$$

where $I_0(z)$ and $I_1(z)$ are the modified Bessel functions. The asymptotic expressions for $u \gg \tau$ have the form

$$J_B^{as}(\tau, u, \lambda) \approx \frac{L}{4}\sqrt{\frac{\lambda}{\pi u}} e^{-(1-\lambda)u - \frac{\lambda \tau^2}{4u}}, \tag{18}$$

$$H_B^{as}(\tau, u, \lambda) \approx \frac{L\tau}{4u}\sqrt{\frac{\lambda}{\pi u}} e^{-(1-\lambda)u - \frac{\lambda \tau^2}{4u}}. \tag{19}$$

When $\lambda = 1$, these expressions also coincide with the expressions (7) and (8) of the diffusion approximation.

First of all, let us consider the case A. When $\lambda = 1$, the exact and approximate values of $J(\tau, u, \lambda)$ and $H(\tau, u, \lambda)$ are pretty close to each other, and the asymptotic expressions for these quantities coincide with the expressions for $J_D(\tau, u, 1)$ and $H_D(\tau, u, 1)$ in the diffusion approximation. The ratios J_A^{as}/J_D and H_A^{as}/H_D are shown in Table 1.

A different situation occurs when $\lambda < 1$. A comparison of the exact values $J_A(\tau, u, \lambda)$ and $H_A(\tau, u, \lambda)$ with the approximate values of these quantities shows that they can significantly differ from each other. The asymptotic expressions differs from the corresponding expressions in the diffusion approximation. Their ratio is equal to

$$\frac{J_A^{as}(\tau, u, \lambda)}{J_D(\tau, u, \lambda)} = \frac{H_A^{as}(\tau, u, \lambda)}{H_D(\tau, u, \lambda)} \approx \frac{1}{\sqrt{\lambda}}\, e^{-\frac{\tau^2}{4u}\left(\frac{1}{\lambda} - 1\right)}. \tag{20}$$

Since λ is included in the exponent, these ratios may differ significantly from the unity.

Let us consider now the case B. We note that due to the finite speed of light $J(\tau, u, \lambda) = 0$ and $H(\tau, u, \lambda) = 0$ if $u < \tau$ but in the diffusion approximation

Table 1: Ratios of J_A^{as}/J_D and H_A^{as}/H_D for $\lambda = 1$

	$\tau = 1$		$\tau = 10$	
u	J_A^{as}/J_D	H_A^{as}/H_D	J_A^{as}/J_D	H_A^{as}/H_D
I	0.801	0.98	5.30×10^7	8.57×10^6
2	0.916	1.57	7.78×10^2	2.24×10^2
3	0.977	1.76	2.92×10^1	1.15×10^1
4	0.995	1.77	6.92	3.33
5	1.018	1.69	3.27	1.82
6	1.022	1.59	2.12	1.32
7	1.024	1.50	1.63	1.104
8	1.023	1.43	1.38	0.997
9	1.022	1.36	1.23	0.942
10	1.021	1.32	1.14	0.914
15	1.015	1.19	0.993	0.906
20	1.012	1.13	0.971	0.939
30	1.008	1.083	0.977	0.983
40	1.006	1.060	0.985	1.001
50	1.005	1.047	0.990	1.009
60	1.004	1.039	0.993	1.012
80	1.003	1.029	0.996	1.014
100	1.002	1.023	0.998	1.014

$J_D(\tau, u, \lambda) \neq 0$ and $H_D(\tau, u, \lambda) \neq 0$ under this condition as the finite speed of light is not taken into account in this approximation. A comparison of the exact and asymptotic expressions gives approximately the same results, as in the case of A, i.e. $J_B^{as}(\tau, u, 1) = J_D(\tau, u, 1)$ and $H_B^{as}(\tau, u, 1) = H_D(\tau, u, 1)$, but when $\lambda < 1$, $J_B^{as}(\tau, u, \lambda)$ and $H_B^{as}(\tau, u, \lambda)$ are significantly different from $J_D(\tau, u, \lambda)$ and $H_D(\tau, u, \lambda)$, as

$$\frac{J_B^{as}(\tau, u, \lambda)}{J_D(\tau, u, \lambda)} = \frac{H_B^{as}(\tau, u, \lambda)}{H_D(\tau, u, \lambda)} \approx \sqrt{\lambda}\, e^{\frac{\tau^2}{4u}(1-\lambda)}, \tag{21}$$

i.e., these ratios depend strongly on λ (see Tables 2 and 3).

From the above it follows that the replacement of the non-stationary radiation transfer equation by the heat conductive equation should give satisfactory results when $\lambda \approx 1$ and can lead to significant errors when $\lambda < 1$.

This conclusion is also valid in the cases of non-stationary radiative transfer in infinite three-dimensional media illuminated by planar or point sources. Let us give the expressions of the mean intensity and radiation flux in these cases (if one uses the Eddington approximation).

Let us consider an infinite medium illuminated by a momentary isotropic planar source which can be represented in the form of multiple isotropic point sources of luminosity L uniformly distributed on the plane $\tau = 0$ with a surface

Table 2: Values of $J_A(\tau,u)$, $J_D(\tau,u)$, $J_B(\tau,u)$ for $\lambda = 0.5$

	$\tau = 1$			$\tau = 10$		
u	$J_A(\tau,u)$	$J_D(\tau,u)$	$J_B(\tau,u)$	$J_A(\tau,u)$	$J_D(\tau,u)$	$J_B(\tau,u)$
I	5.50×10^{-2}	6.66×10^{-2}	3.32×10^{-2}	2.84×10^{-5}	1.19×10^{-12}	0
2	3.24×10^{-2}	3.24×10^{-2}	1.82×10^{-2}	3.82×10^{-5}	1.37×10^{-7}	0
3	1.88×10^{-2}	1.67×10^{-2}	1.01×10^{-2}	4.11×10^{-5}	4.37×10^{-6}	0
4	1.09×10^{-2}	8.96×10^{-3}	5.63×10^{-3}	3.91×10^{-5}	1.84×10^{-5}	0
5	6.30×10^{-3}	4.92×10^{-3}	3.18×10^{-3}	3.43×10^{-5}	3.49×10^{-5}	0
6	3.64×10^{-3}	2.75×10^{-3}	1.82×10^{-3}	2.84×10^{-5}	4.44×10^{-5}	0
7	2.11×10^{-3}	1.55×10^{-3}	1.04×10^{-3}	2.25×10^{-5}	4.53×10^{-5}	0
8	1.22×10^{-3}	8.85×10^{-4}	5.94×10^{-4}	1.73×10^{-5}	4.01×10^{-5}	0
9	7.08×10^{-4}	5.08×10^{-4}	3.43×10^{-4}	1.29×10^{-5}	3.25×10^{-5}	0
10	4.11×10^{-4}	2.93×10^{-4}	1.99×10^{-4}	9.37×10^{-6}	2.47×10^{-5}	7.78×10^{-5}
15	2.82×10^{-5}	1.98×10^{-5}	1.36×10^{-5}	1.51×10^{-6}	3.80×10^{-6}	6.95×10^{-6}
20	2.02×10^{-6}	1.41×10^{-6}	9.81×10^{-7}	1.90×10^{-7}	4.10×10^{-7}	5.79×10^{-7}
30	1.11×10^{-8}	7.81×10^{-9}	5.46×10^{-9}	2.09×10^{-9}	3.42×10^{-9}	3.78×10^{-9}
40	6.50×10^{-11}	4.57×10^{-11}	3.20×10^{-11}	1.81×10^{-11}	2.46×10^{-11}	2.41×10^{-11}
50	3.92×10^{-13}	2.76×10^{-13}	1.93×10^{-13}	1.40×10^{-13}	1.68×10^{-13}	1.54×10^{-13}
60	2.41×10^{-15}	1.70×10^{-15}	1.19×10^{-15}	1.02×10^{-15}	1.12×10^{-15}	9.82×10^{-16}
80	9.47×10^{-20}	6.68×10^{-20}	4.70×10^{-20}	4.99×10^{-20}	4.90×10^{-20}	4.06×10^{-20}
100	3.85×10^{-24}	2.71×10^{-24}	1.91×10^{-24}	2.32×10^{-24}	2.12×10^{-24}	1.70×10^{-24}

density of l and flashing at the initial moment of time ($u = 0$). Then, using the diffusion approximation, we have

$$J_D(\tau,u) = \frac{lL}{8\pi\sqrt{\pi}}\,\frac{\sqrt{3-x_1}}{\sqrt{u}}\,\exp\left(-\frac{(3-x_1)\,\tau^2}{4u}\right), \tag{22}$$

$$H_D(\tau,u) = \frac{lL}{4\sqrt{\pi}}\,\frac{\sqrt{3-x_1}}{u\sqrt{u}}\,|\tau|\,\exp\left(-\frac{(3-x_1)\,\tau^2}{4u}\right), \tag{23}$$

when $\tau \gg 1$, $1-\lambda \ll 1$, $u > \sqrt{3-x_1}\beta_2\tau$.

In the case of an infinite medium illuminated by a momentary point source of luminosity L we have

$$J_D(\tau,u) = \frac{L\alpha^2}{32\pi^2\sqrt{\pi}}\,\frac{(3-x_1)^{\frac{3}{2}}}{u\sqrt{u}}\exp\left(-\frac{(3-x_1)\,\tau^2}{4u}\right), \tag{24}$$

$$H_D(\tau,u) = \frac{L\alpha^2}{16\pi\sqrt{\pi}}\,\frac{(3-x_1)^{\frac{3}{2}}}{u^2\sqrt{u}}\,\tau\,\exp\left(-\frac{(3-x_1)\,\tau^2}{4u}\right), \tag{25}$$

when $\tau \gg 1$, $1-\lambda \ll 1$, $u > \sqrt{3-x_1}\beta_2\tau$.

Table 3: Values of $H_A(\tau, u)$, $H_D(\tau, u)$, $H_B(\tau, u)$ for $\lambda = 0.5$

	$\tau = 1$			$\tau = 10$		
u	$H_A(\tau, u)$	$H_D(\tau, u)$	$H_B(\tau, u)$	$H_A(\tau, u)$	$H_D(\tau, u)$	$H_B(\tau, u)$
I	8.69×10^{-2}	6.66×10^{-2}	7.38×10^{-3}	4.96×10^{-5}	1.19×10^{-11}	0
2	4.12×10^{-2}	1.62×10^{-2}	3.57×10^{-3}	6.18×10^{-5}	6.84×10^{-7}	0
3	1.96×10^{-2}	5.57×10^{-3}	1.75×10^{-3}	6.26×10^{-5}	1.46×10^{-5}	0
4	9.38×10^{-3}	2.24×10^{-3}	8.73×10^{-4}	5.64×10^{-5}	4.61×10^{-5}	0
5	4.52×10^{-3}	9.85×10^{-4}	4.41×10^{-4}	4.71×10^{-5}	6.98×10^{-5}	0
6	2.20×10^{-3}	4.58×10^{-4}	2.26×10^{-4}	3.73×10^{-5}	7.41×10^{-5}	0
7	1.08×10^{-3}	2.22×10^{-4}	1.17×10^{-4}	2.83×10^{-5}	6.47×10^{-5}	0
8	5.37×10^{-4}	1.11×10^{-4}	6.12×10^{-5}	2.08×10^{-5}	5.02×10^{-5}	0
9	2.69×10^{-4}	5.64×10^{-5}	3.24×10^{-5}	1.49×10^{-5}	3.61×10^{-5}	0
10	1.37×10^{-4}	2.93×10^{-5}	1.73×10^{-5}	1.04×10^{-5}	2.47×10^{-5}	8.64×10^{-5}
15	5.37×10^{-6}	1.32×10^{-6}	8.39×10^{-7}	1.41×10^{-6}	2.54×10^{-6}	4.78×10^{-6}
20	2.60×10^{-7}	7.07×10^{-8}	4.63×10^{-8}	1.52×10^{-7}	2.05×10^{-7}	2.91×10^{-7}
30	8.66×10^{-10}	2.60×10^{-10}	1.75×10^{-10}	1.27×10^{-9}	1.14×10^{-9}	1.25×10^{-9}
40	3.63×10^{-12}	1.14×10^{-12}	7.79×10^{-13}	8.74×10^{-12}	6.15×10^{-12}	5.60×10^{-12}
50	1.71×10^{-14}	5.51×10^{-15}	3.79×10^{-15}	5.57×10^{-14}	3.36×10^{-14}	3.04×10^{-14}
60	8.62×10^{-17}	2.83×10^{-17}	1.95×10^{-17}	3.43×10^{-16}	1.87×10^{-16}	1.62×10^{-19}
80	2.50×10^{-21}	8.35×10^{-22}	5.80×10^{-22}	1.27×10^{-20}	6.13×10^{-21}	5.04×10^{-21}
100	8.02×10^{-26}	2.71×10^{-26}	1.89×10^{-26}	4.71×10^{-25}	2.12×10^{-25}	1.63×10^{-25}

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