V.V. Sobolev and Analytical Radiative Transfer Theory

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The review of Sobolev's publications on the analytical radiative transfer theory is presented. A short review is also given of the results published by his disciples.

1 Introduction

The basic equations of the Radiative Transfer Theory (RTT) were formulated at the turn of the 20th century. Initially transfer theory developed as a purely analytical instrument since the calculation of radiation fields was problematic with the computational facilities of that time. Between 1940 and 1980 the exact and sufficiently accurate approximate solutions to the basic equations of the theory were found, and for various limiting cases the asymptotic theory was developed. The peculiarities and difficulties of the description and computation of multiple scattering were thus revealed. By comparing numerical results with the analytical solutions it became possible to evaluate the benefits and drawbacks of various numerical methods and give estimates of their accuracy.

V.V. Sobolev made a definitive contribution to the creation of the analytical RTT. In the 1940s he developed a method of calculating populations of atomic levels in expanding non-planar dilute gaseous media, the method which is still in use. This method is known now as *the Sobolev theory*. He also developed an effective approximate method to solve problems of anisotroping multiple light scattering. As early as in 1941, he formulated the approximation of Complete Frequency Redistribution (CFR) in problems of radiative transfer in spectral lines. In the 1950s he developed the method of exact solution of the basic integral equations describing multiple light scattering, both monochromatic and with CFR. He was also the first to investigate multiple scattering of polarized radiation and non-stationary radiation fields. He applied his theoretical findings to the interpretation of observations of many types of astrophysical objects. Dozens of former Sobolev's students form a team of theorists known as the Sobolev astrophysical school. In what follows we present a brief review of the main results found by V.V. Sobolev and his disciples.

We begin with the description of contributions to the analytical RTT by V.A. Ambartsumian (who was Sobolev's Ph.D. adviser).

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2 Contribution of V.A. Ambartsumian

V.A. Ambartsumian founded the chair of astrophysics (1934) in the St. Petersburg (Leningrad) University. He published the first Russian manual on theoretical astrophysics [1].

He studied the radiation regime in an infinite plane medium with sources at the infinite depth, thus modeling deep layers of a semi-infinite medium with anisotropic monochromatic scattering [2].

Ambartsumian revealed the important role of radiation pressure by spectral line photons in the dynamics of planetary nebulae and stellar envelopes, particularly, the pressure exerted by the photons of the hydrogen L_α-line [3]. He suggested a new method to describe the influence of absorption lines on the temperature regime in stellar atmospheres [4].

Ambartsumian introduced innovative approaches to RTT problems known as the invariance principles and the method of adding of layers [5]. Using these new methods he expressed the reflection and transmission coefficients of a plane layer which are functions of two angular variables in terms of auxiliary functions of one variable [6]. For these functions he found nonlinear integral equations and studied the asymptotic behavior of their solutions for the case of a layer of large optical thickness [7, 8].

He expressed the mean number and the mean square of the number of scattering events in terms of the radiation intensity [9].

He studied also the problem of light scattering in semi-infinite medium with reflecting surface [10].

The main Ambartsumian's publications on RTT are reprinted in the book [11]. The proceedings of the conference dedicated to the 40th anniversary of the Invariance Principle are published in [12].

V.A. Ambartsumian studied many other astrophysical problems: the lifetimes of stars, star clusters, stellar associations, the Milky Way brightness fluctuations, formation of galaxies, variable stars, etc.

V.V. Sobolev continued studies of his teacher in RTT. He discovered new branches and created new methods of RTT, formulated and solved a lot of new problems.

3 Early Sobolev's publications

V.V. Sobolev proposed a method of approximate solution to the problem of anisotropic scattering of monochromatic radiation. According to this method the first scattering is taken into account exactly, with the real phase function, whereas higher order scatterings are treated approximately, with the two-term phase function [13]. V.V. Sobolev applied the developed theory to terrestrial and planetary atmospheres [14, 15]. Later, this approximate method was applied to problems with spherical geometry: scattering in a homogeneous sphere with a point source at its center [16] (a model of dust nebula) and in a spherical shell illuminated by a parallel radiation flux [17] (model of a planetary atmosphere, coauthor I.N. Minin).

V.V. Sobolev showed that the idea of accelerated expansion of planetary nebulae adopted at that time is incorrect because it was based on the assumption that line radiation does not change its frequency when scattered. In fact, the scattered photon reduce its frequency because a part of its momentum and energy passes to the scattering atom. Therefore the full momentum of stellar radiation is not transmitted to the nebula matter and does not accelerate it: radiation simply leaves the nebula in the wings of the line [18]. In [19] V.V. Sobolev simplified the calculation of the radiation regime in infinite plane medium.

The problems of radiative transfer in expanding media were studied in Sobolev's doctoral thesis and in his famous book [20]. The equations determining the populations of atomic levels were derived and solved using the method of local scattering. The method is known as *the Sobolev approximation* and is widely used till now. The essence of the method is the following. If a medium expands with a velocity gradient, the radiation in a line ceases to interact with atoms when it propagates in places where gas velocity is substantially different from the velocity at the site of its emission. As a result, the line radiation is not reabsorbed and propagates freely. The scattering becomes local. Due to this effect, in media moving with large gradient of gas velocity atomic excitation and degree of ionization change drastically [21].

Later, for the special case of the two-level atom and a constant velocity gradient in plane media, the integral equation was formulated, with the kernel depending on the absolute value of the difference of the arguments. The approximate solution of the equation was found using "on the spot" approximation [22].

In two papers [23] and [24] (with V.V. Ivanov) the intensities of hydrogen lines and the Balmer decrement in the spectra of hot stars were calculated. Lines are formed in their envelopes. By applying the approximation of local scattering, the equations governing the populations of atomic levels were reduced to algebraic ones.

4 Monochromatic scattering

4.1 Polarized radiation and non-stationary radiation fields

In [25] V.V. Sobolev formulated transfer equations for linearly polarized radiation for the case of Rayleigh scattering. He found the behavior of two intensities and the corresponding source functions in deep layers of semi-infinite medium. He also found the degree of polarization of the radiation emerging from purely electronic semi-infinite medium with the sources at infinite depth, thus modeling a hot atmosphere of an early type star. The largest degree of polarization, 11.7%, is reached at the limb of the stellar disk. This is known as the *Sobolev–Chandrasekhar polarization limit.*

Later V.V. Sobolev published several papers on Rayleigh scattering (with V.M. Loskutov). They calculated fields of polarized radiation in plane slabs for several distributions of primary sources [26]. The results were used for the interpretation of observed polarization of X-ray sources [27] and quasars [28].

For studying non-stationary radiation fields in stationary media V.V. Sobolev introduced two characteristic times [29]: t_1, the mean time a photon spends while absorbed by an atom and t_2, the mean time between two consecutive scatterings of a photon. He derived the equations describing non-stationary radiation fields in one-dimensional approximation and solved them for the case $t_2 = 0$, both for final and infinite optical thickness of the medium. The solutions thus found were used to interpret peculiarities of radiation fields in the ejecta of novae (see [30]).

Later on V.V. Sobolev continued studying the non-stationary scattering with his coauthor A.K. Kolesov [31, 32]. They presented the formulas and numerical data for the solutions to the problem of a point source in an infinite and semi-infinite one-dimensional media for alternative cases $t_2 \ll t_1$ and $t_1 \ll t_2$. The results were applied to interpret the flares of UV Ceti stars.

4.2 Reflecting boundaries and inhomogeneous media

V.V. Sobolev derived the equations for radiation fields in a plane media with a reflecting lower surface. Two particular cases were considered in more detail: orthotropic and mirror reflection. In the former case the all quantities with the reflecting surface were expressed by simple relations in terms that without it [33]. The results for the case of a mirror boundary were published in the book [30] and applied to the scattering in a cloudy slab of large optical thickness above the surface of the sea.

The problem of scattering in plane media if the probability of photon survival λ depends on the depth τ was considered [34, 35]. The calculations of the albedo and brightness coefficients were made for the cases:

1) λ is piecewise constant;
2) λ is an exponent of optical depth $\lambda = \lambda_0 \, e^{-m\tau}$ or the sum of such exponents;
3) λ is a superposition (integral) of exponents.

Later the degree of polarization of the radiation emergent from the semi-infinite medium was calculated for the case 2) [36] (coauthor V.M. Loskutov).

4.3 New methods of calculation of radiation fields

V.V. Sobolev formulated the concept of photon escape probability from a medium: the product $2\pi p\,(\tau, \eta)\,\mathrm{d}\eta$ denotes the probability for a photon absorbed at a depth τ in isotropically scattering semi-infinite atmosphere to escape from this medium at an angle $\arccos \eta$ within a solid angle of $2\pi\,\mathrm{d}\eta$ after an arbitrary number of scatterings. It is easy to obtain the equations and relations for the escape probability from simple considerations. If this function is found, it is possible with the known power of primary sources to calculate the intensity of emergent radiation by direct integration [30]. Apart from this, the majority of the functions and equations of RTT got the probability interpretation. The concept of the escape probability was applied to many problems of RTT for deducing the equations and solving them.

Another method which was applied by V.V. Sobolev is transformation from equations with integrals on optical depth τ to linear equations with integrals on angular variables which is equivalent to application of the Laplace transform. Such equations were derived for brightness coefficients, functions of one variable in terms of which these coefficients were expressed and other functions. It was such type equations that were used for the calculations of polarization fields with the Rayleigh scattering and were mentioned above.

4.4 Asymptotic theory of monochromatic scattering

The complete asymptotics of the source function and of the intensity in deep layers of a semi-infinite medium for the reflection problem were obtained by V.V. Sobolev (see [37]) using the relations between characteristics of anisotropic scattering in infinite and semi-infinite media which were found with the summation of layers method. Using these results and with the same method V.V. Sobolev deduced asymptotics for the brightness coefficients and other functions when the optical thickness of a slab τ_0 was large [38].

Another domain for which the asymptotic formulas were found is a nearly pure scattering when the survival probability is very close to unity: $1 - \lambda \ll 1$. Expansions of various functions on the power of $\sqrt{1-\lambda}$ (the first or second) were obtained. The results are given in the book [39].

5 Scattering in lines and the resolvent method

5.1 Frequency redistribution

V.V. Sobolev directed essential efforts to the study of scattering in spectral lines.

The laws describing the transformation of photon frequency in single scattering were deduced but they were too complicate and did not allow to solve the problem of line formation. Several authors (T. Holstein, L.M. Biberman, V.V. Sobolev, and others) proposed the approximation of complete redistribution in frequency (CFR), which implied that the photon frequencies before and after scattering do not correlate. In other words, the absorption and emission coefficients depend on frequency equally. The following additional approximations were accepted: atoms of the same kind have only two discrete levels (the two-level approximation) and continuum constant within the line; both radiative and collisional transitions are possible between these levels; the induced radiation was not taken into account because it leads to nonlinear equations, which do not permit analytical investigation.

To begin V.V. Sobolev derived some frequency redistribution laws and accepted as the approximation the CFR. Then he considered scattering in a one-dimensional medium and obtained differential equations for the intensity and integral equation for the source function. He solved them for homogeneous distribution of the sources and found the emissivity, the density and flux of energy, the light pressure and the emission profiles of emergent radiation. The results were close for various redistribution laws and strongly differed from the monochromatic scattering.

Then various equations were obtained for the probability $p(\tau, x_1, x_2)$ of photon escape from the medium of optical thickness τ_0 in line from the depth τ (x_1 and x_2 are dimensionless frequencies of the emitted and escaping photons). Also the equations were derived for the two introduced functions $\varphi(x, \tau_0)$ and $\psi(x, \tau_0)$. After that V.V. Sobolev found the equation for the brightness coefficients. The equations were solved for CFR and the profiles of the forming absorption lines were calculated. Better agreement with the observable ones than for monochromatic scattering were achieved. All these results are in his book [30].

The integro-differential and integral equations, describing the process of multiple photon scattering in spectral line in a plane layer on the assumption of CFR, were derived by V.V. Sobolev in [40]. The approximate solution of the integral equation based on the principle of local scattering was found. It is usually known as on the spot approximation. Later V.V. Sobolev developed the exact theory of multiple scattering known as resolvent method. At first it was done for isotropic monochromatic scattering [41, 42] and then for scattering in line with CFR [43, 44].

5.2 Resolvent method

This method is applicable to equations of the following form:

$$S(\tau) = S_0(\tau) + \frac{\lambda}{2} \int_{\tau_*}^{\tau_0} K(|\tau - \tau'|) S(\tau')\, d\tau'. \tag{1}$$

This equation is the basic integral equation of RTT. Here $S_0(\tau)$ is a given, and $S(\tau)$ is the sought-for source function, λ is photon survival probability per scattering. The limits of integration τ_* and τ_0 are the "depths" of the lower and upper boundaries of a plane medium. If $-\tau_* = \tau_0 = \infty$, the medium is infinite; if $\tau_* > -\infty$ and $\tau_0 = \infty$, the medium is semi-infinite; if $\tau_0 < \infty$, it is a finite plane slab. In the last two cases it can be assumed that $\tau_* = 0$. The kernel function $K(\tau)$ for both monochromatic and CFR scattering can be represented as a superposition (integral) of exponentials.

The resolvent is defined as a function that allows one to find the solution of Eq. (1) for arbitrary given $S_0(\tau)$

$$S(\tau) = S_0(\tau) + \int_{\tau_*}^{\tau_0} \Gamma(\tau, \tau') S_0(\tau')\, \mathrm{d}\tau'. \tag{2}$$

The notations of resolvents are as follows: for an infinite medium it is $\Gamma_\infty(\tau,\tau_1)$, for a semi-infinite medium $\Gamma(\tau,\tau_1) = \Gamma(\tau,\tau_1,\infty)$, and for a finite slab $\Gamma(\tau,\tau_1,\tau_0)$.

For an infinite medium the following obvious relation holds: $\Gamma_\infty(\tau,\tau_1) = \Gamma_\infty(|\tau-\tau_1|,0) \equiv \Phi_\infty(|\tau-\tau_1|)$. V.V. Sobolev has shown that the resolvent of the equation (1) can be expressed in terms of a function of one variable, namely, the particular value of the resolvent with one of its arguments set equal to 0. This function is called the *resolvent function*: $\Phi(\tau,\tau_0) \equiv \Gamma(\tau,0,\tau_0)$. If $\tau_* = 0$, the explicit expression of Γ in terms of Φ is rather complicated

$$\Gamma(\tau,\tau_1,\tau_0) = \Phi(|\tau-\tau_1|,\tau_0) + \int\limits_0^{\min(\tau,\tau_1)} \big[\Phi(\tau-t,\tau_0)\Phi(\tau_1-t,\tau_0) - \Phi(\tau_0-\tau+t,\tau_0)\Phi(\tau_0-\tau_1+t,\tau_0)\big]\,dt. \tag{3}$$

For semi-infinite medium one has to set $\tau_0 = \infty$ and $\Phi(\infty,\infty) = 0$.

For the kernel functions representable as a superposition of exponentials V.V. Sobolev derived linear and nonlinear equations for the Laplace transforms of the resolvents and the resolvent functions as well as equations for resolvent functions themselves of type (1) (with $S_0(\tau) = (\lambda/2)\,K(\tau)$) and of Volterra type. Some of these equations are generalizations of Ambartsumian's equations. For a finite slab alternative equations were derived, with the derivatives with respect to τ_0.

For isotropic monochromatic scattering V.V. Sobolev found the asymptotic form of $\Phi(\tau,\tau_0)$ for $\tau \gg 1$[45]. It is expressed in terms of the resolvent function of semi-infinite medium. For the latter the exact explicit expression is known.

V.V. Sobolev encouraged his pupils for further development of the theory of line formation with CFR.

5.3 Inhomogeneous, infinite and spherical media

The scattering theory in inhomogeneous media was extended to the scattering in a spectral line with CFR [46] taking into account continuous absorption. In [47] V.V. Sobolev and E.G. Yanovitsky applied the resolvent method to the case of scattering with variable $\lambda(\tau)$. In [48] the results for the variable λ were summarized.

In [49] it was shown that three problems of monochromatic isotropic scattering in three media, namely: in a semi-infinite medium with an ideally reflecting mirror boundary; in a stationary spherical shell geometrically thin but optically thick with the central source and also in an infinite medium with a point source, are reduced to scattering in infinite medium.

In [50] the case of a smoothly reflecting boundary was reduced to two integral equations and with two resolvent functions. In [51] along with the fact that the smooth reflection from the boundary is not ideal, the changing direction of radiation when crossing it was taken into account because the refraction indexes differ on its two sides. For the problem of diffuse reflection the equations were obtained for the azimuthal harmonics of the reflection coefficient by assuming that such harmonics calculated without reflection from the boundary are known. Also, the equations were deduced for characteristics of emergent radiation in the Milne problem and of the regime of the radiation field in deep layers. These problems are to model the light scattering at sea.

Several papers were devoted to monochromatic scattering in a homogeneous sphere and in a spherical envelope. In [52] the problem of scattering in the sphere with spherically symmetric sources was reduced to the problem of a plane slab of double optical thickness. In [53] the asymptotic formulas were obtained for the intensity of emergent radiation $I(\eta, \tau_0)$ when there is a point source in the center of the sphere or in the center of a thin spherical envelope and when the optical thicknesses of the sphere and the envelope τ_0 are large. V.V. Sobolev and A.K. Kolesov found more exact asymptotics of $I(\eta, \tau_0)$ for illuminating a sphere both by a radiation flux [54] and by a point source in the center [55]. The summary of these researches was presented in [56].

5.4 The resolvent method for anisotropic scattering

The equations governing anisotropic scattering with an arbitrary phase function contain integrals over three variables τ, η and ϕ. It is possible to expand characteristics of scattering in the Fourier series (or finite sum) on azimuth ϕ, to separate azimuth harmonics and to deduce separate equations with double integrals for each of the harmonics.

The intensity of the emergent radiation in the problem of reflection and transmission may be expressed in terms of functions $\varphi_i^m(\eta, \tau_0)$ and $\psi_i^m(\eta, \tau_0)$. If the number of terms in the expansion of the phase function in the Legendre polynomials equals $n+1$, to get the harmonic with number m $(i = m, m+1, \ldots, n)$ one has to find $2(n - m + 1)$ such functions. For semi-infinite medium the functions $\psi_i^m(\eta, \infty) = 0$. For semi-infinite medium V.V. Sobolev expressed all the functions $\varphi_i^m(\eta)$ for each of the harmonics in terms of one function $H^m(\eta)$. The functions $\varphi_i^m(\eta, \tau_0)$ and $\psi_i^m(\eta, \tau_0)$ were expressed in terms of two functions, $X^m(\eta, \tau_0)$ and $Y^m(\eta, \tau_0)$. The polynomials depending on η and λ entered these expressions as factors. They are given by recurrent relations. The resolvent for each of the harmonics is expressed in terms of one resolvent function $\Phi_m(\tau, \tau_0)$. These results are summarized in Sobolev's book [39].

6 Other Sobolev's works on RTT

6.1 Number of scatterings; strongly peaked phase function

In four of V.V. Sobolev's papers [57] the numbers of scatterings were expressed through the functions introduced in other works. In the fourth paper for the case of scattering in the spectral line with CFR in finite slab were obtained sufficiently narrow upper and lower estimations of the number of scatterings for large optical thickness.

In the case of strongly elongated forward phase function V.V. Sobolev expanded the intensity according the Taylor formula of the second order in the powers of the difference between the polar angles of the scattered and the incident radiation, and replaced the integral over angles with the differential operator. With the help of the obtained equation he found the light regime in deep layers [58].

6.2 Scattering in planetary atmospheres

Twenty years after publishing [14] V.V. Sobolev resumed the study of scattering characteristics in the Venus atmosphere. In the first paper [59] he calculated the reflection coefficient for a two-term phase function with the terms proportional to $\sqrt{1-\lambda}$. In the second paper two models of the atmosphere were adopted: the homogeneous one consisting of molecules and large-grained particles; and the two-layer one in which a molecular slab is placed above a cloudy slab. The dependencies of the degree of polarization on the phase and wavelength were found.

In two papers [60, 61] (coauthors I.N. Minin) the radiation of planetary atmosphere was described for isotropic scattering. The atmosphere was assumed to be plane (with the dependence $\lambda(\tau)$), but the incident angles of solar radiation on the plane were chosen to be the same as those of a parallel flux on spherical atmosphere. The effect of orthotropic reflection from the surface was taken into account.

In [62] the formulas for the profile r_ν and equivalent width W of a line in a certain place of the planetary disk and from the whole disk were derived as functions of phase. In [63] a two-layer atmosphere was constructed of a semi-infinite medium and an optically thin slab above it with different optical properties (i.e. their phase functions and photon survival probabilities differed). The same formulas were obtained.

6.3 Emission of supernovae and electron scattering

In three papers [64, 65, 66] V.V. Sobolev (in the third coauthor A.K. Kolesov) calculated the continuous spectra, light curves, optical thickness of envelopes and spectrophotometric temperatures of supernovae on the early stages of expansion of the ejecta. It was adopted that the radiation of envelope was under strong effect of electron scattering.

The effect of electron scattering on the spectra of stationary hot stars was studied in [67, 68], which continued the study in [35]. The emergent flux and specrophotometric temperature were calculated using as a tool linear integral equations with the integrals on angular variable.

6.4 Global absorption and emission

V.V. Sobolev devoted several papers specially to determination of relation between two parts of radiation energy that enters into a scattering and absorbing medium. One part of this energy undergoes true absorption and transfers to other types of energy. The other part abandons the medium. In the most general form the problem was considered in [69]. The law of redistribution in frequency and direction as the fraction of reemitted photons could be different in different points of the medium of arbitrary form (in [70] scattering was supposed to be isotropic). The amount of energy absorbed from the flux illuminating the medium was shown to be connected with the amount of irradiated energy if the distribution of internal sources was uniform. Analogous relations were obtained for Rayleigh scattering of polarized radiation [71].

The usefulness of the obtained relation was demonstrated for isotropic monochromatic scattering as well as for scattering in a line with CFR in a semi-infinite medium, in a plane slab and in a homogeneous sphere [72]. V.V. Sobolev deduced the integral relations for the intensities of internal and emerging from a plane slab radiation in [73].

7 Contributions of Sobolev's disciples

Here we present a list of the main achievements made by Sobolev's students and disciples. More detailed reviews of their works and the works of other authors are presented in the symposiums proceedings [12, 74, 75, 76].

7.1 I.N. Minin

The papers published with V.V. Sobolev as a coauthor are [17, 60, 61].

I.N. Minin deduced the equation and proposed the method to calculate the radiation transfer in a medium with refraction [77, 78] and obtained the exact expression for the resolvent function for monochromatic isotropic scattering in semi-infinite medium [79].

I.N. Minin used the Laplace transforms on time for solving the non-stationary radiation transfer in a medium with monochromatic scattering and studied it in detail [80, 81]. He showed that in three particular cases when $t_2 = 0$, $t_1 = 0$ and $t_1 = t_2$ the solutions for $\lambda < 1$ can be expressed through the solutions for $\lambda = 1$. The exact and asymptotic formulas for characteristics of the radiation emerging from a finite one-dimensional medium were obtained in three mentioned cases [82].

In [83] many characteristics of radiation field in a semi-infinite medium with arbitrary values of t_1 and t_2 were expressed in terms of one function. The equation determining this function was derived. Time-dependent problems were also solved for non-stationary one-dimensional ($\tau(t) = \tau(0)e^{-\alpha t}$) [84] and inhomogeneous [85] media.

Anisotropic scattering in semi-infinite medium [86] and in the layer of finite optical thickness [87] was investigated. If the number of terms in the expansion of phase function in the Legendre polynomials is equal to $n+1$, then for azimuthal harmonic number m I.N. Minin introduced $(n+1-m)^2$ resolvent functions and derived equations for them. His results are in his reviews [88, 89] and book [90].

7.2 V.V. Ivanov

The derivation of various asymptotics that characterize scattering in a spectral line with CFR directly from the equations [91, 92].

The wide use of the concept of thermalization length τ_{t} (depending on λ). Its value separates two regions. In the first one (depths $\tau < \tau_{\mathrm{t}}$) the scattering in line can be considered as conservative while in the second one ($\tau > \tau_{\mathrm{t}}$) photons are thermalized, i.e. the source function becomes proportional to the Planck function [93]. Asymptotic formulas for the resolvent functions $\Phi_\infty(\tau)$ and $\Phi(\tau)$ for $\tau \gg 1$ depend not τ and λ separately but in the essential parts only on τ/τ_{t}.

Time-variations of the degree of excitation for two-level atoms and of the line profile formed in an infinite homogeneous medium with CFR and $t_1 = 0$ if initially the atoms are completely excited [94]. The leading terms of the asymptotics at large time intervals coincide with those obtained for a more exact law of redistribution.

The detailed description of the asymptotic theory of conservative scattering [95, 96] was made for the Milne problem with isotropic monochromatic scattering as well as for CFR scattering with the absorption coefficient decreasing in the line wings as a power of frequency. The asymptotics of X and Y-functions were expressed in terms of the Bessel functions. For the Doppler profile it was performed earlier [97]. It was the very first result of what is now known as the large-scale description.

The concept of the "mean length of a photon path" $\overline{T}$, i.e. path from the place of photon emission to the place where it is finally absorbed (i.e. thermalized) was introduced and the formulas determining $\overline{T}$ were derived [98].

The formulation of a high accuracy approximate solution [99] to the basic CFR integral equation of RTT, both for half-space and for plane layer of finite thickness.

The description of multiple scattering of spectral line photons as a stochastic process of the Lévy random walks was given. It was used to obtain various asymptotics of CFR RRT (with Sh.A. Sabashvili) [100].

The solution was found to the problem of diffuse reflection and transmission of radiation if $t_1 = 0$, $t_2 = 1$ and the boundary of anisotropically scattering layer of finite optical thickness τ_0 is illuminated by an instant light flash [101] (with S.D. Gutshabash). The asymptotic behavior of the brightness wave escaping the layer was found assuming the thickness of the layer $\tau_0 \gg 1$.

The process of frequency relaxation to CFR due to multiple scatterings with non-CFR redistribution functions was studied [102] (with A.B. Schneeweis).

Generalizations of the invariance principles for a semi-infinite medium with scalar anisotropic scattering [103] and for scattering of polarized radiation were formulated [104] (with H. Domke). The asymptotic forms of the basic functions were found explicitly.

New concepts for treating analytically the so called blanketing effect were introduced: the "partial intensity", i.e. the contribution to the intensity by photons classified both by the value of the absorption coefficient and by the frequency, and the so-called "gray in the average" atmosphere, in which the opacity probability distribution function (OPDF) is the same along the whole spectrum. The equations describing the radiation transfer in such an atmosphere were given [105] (with A.G. Kheinlo).

Molecular and Rayleigh scattering of polarized radiation was studied in detail using the concept of matrix transfer equation (with V.M. Loskutov, S.I. Grachev, and T. Viik). In particular, the so called $\sqrt{\varepsilon}$ law of the scalar theory was generalized to incorporate polarization [106, 107, 108].

V.V. Ivanov with coauthors investigated scattering polarization of radiation in resonance lines under the assumption that angular and CFR frequency redistributions are not correlated [109].

The albedo shifting method when the kernel of the integral equation is changed to another one in order to accelerate the convergence of iterations was developed (with coauthors) [110, 111, 112].

Ivanov's results are published also in books [113, 114] and in review [115].

7.3 A.K. Kolesov

Articles with V.V. Sobolev [31, 32, 54, 55, 66].

In three papers [116, 117, 118] calculations were made for the Henyey–Greenstein phase function.

In the series of papers [119, 120, 121, 122] the radiation fields in two-layer and multilayer media with anisotropic scattering in the layers were studied. In the most general case the layers differed in the values of λ, phase functions and refraction indexes.

The expansions in the elementary solutions of the radiation transfer equation (the Case method) were applied for radiation fields in non-plane media with anisotropic scattering. In [123] the problem of scattering in a homogeneous sphere was reduced to the plane one. The same procedure was made for a point source in an infinite medium [124]. In [125] the expression for the Green function of the point source and in [126] the asymptotics of this function were obtained, and in [127] the intensity of radiation far from the point source in the infinite medium was expanded in the reversed powers of τ. The case of small true absorption was considered separately. In [128] and [129] the Case modes were found and the relations of their orthogonality were formulated for spherical and cylindrical symmetries.

Review [130].

7.4 E.G. Yanovitskij

With V.V. Sobolev [47].

Detailed investigation of anisotropic scattering in inhomogeneous [131] and multilayer media [132, 133, 134] (the first and the third papers with Zh.M. Dlugach).

Some formulas for the pure scattering were shown to coincide for an arbitrary scattering phase function [135].

For a semi-infinite medium [136] and a plane layer [137] the equations were formulated, which have the form of the transfer equation in the so-called pseudo problems of anisotropic scattering. These equations determine the intensity of radiation, which would correspond to the source functions equal to the resolvent functions $\Phi^m(\tau)$ and $\Phi^m(\tau, \tau_0)$ that were introduced by V.V. Sobolev.

A new form of the radiation transfer equation (called Q-form) was deduced, where the intensity was represented as the derivative on the optical depth of some linear integral operator of the same intensity [138].

The results of Yanovitskij and his coauthors can be found in his book [139].

7.5 D.I. Nagirner

Using the methods of the theory of complex variables the exact explicit solutions and their asymptotic forms were obtained for stationary [140, 141] and non-stationary ($t_2 = 0$) [142] multiple scattering with CFR in infinite and semi-infinite media. Large-scale and uniform asymptotics of the resolvent and other functions describing scattering in a plane layer [143] and sphere [144] of large optical thickness and radius were found, in particular, the mean number of scatterings and dispersion.

The exact and asymptotic formulas describing the process of damping of the atomic excitation in a homogeneous infinite medium, the excitation being created instantly at the initial moment [145]. The scattering in a line with CFR and the Lorentz absorption profile with an arbitrary ratio of t_1 and t_2 parameters was assumed.

The method was proposed to calculate the eigenvalues and eigenfunctions of the basic integral equation (continuous $\lambda(u) = 1/V(u)$ for a semi-infinite medium and discrete $\lambda_n(\tau_0) = 1/V(u_n(\tau_0))$ for a layer of finite thickness). The asymptotics (on n and $\tau_0 \gg 1$) of the eigenvalues were found for an optically thick layer [146].

The resolvent function, its asymptotic behavior and the asymptotics of the spectrum of the basic integral equation for a cylinder were obtained [147].

The method to calculate the scattering in a plane layer of finite optical thickness was proposed [148].

Reviews [149, 150, 151, 152, 153] and books [154, 155].

7.6 V.M. Loskutov

With V.V. Sobolev [26, 27, 28, 36] and V.V. Ivanov [108, 109].

With a given value of the characteristic number k the value of λ is found by the expansion in a chain fraction for the Henyey–Greenstein phase function [156].

The full phase matrix for the Rayleigh scattering is represented by the product $\mathbf{A}(\eta,\phi)\mathbf{A}^{\mathrm{T}}(\eta',\phi')$, where $\mathbf{A}(\eta,\phi)$ is a matrix of the size 3×6. This representation separates the angular variables of the incident (η',ϕ') and scattered (η,ϕ) radiation. For the six-term vector of the source functions the system of integral equations was obtained. Within the same approximation, in which redistribution depends on frequency and on angles independently as in [109], the matrix equation for the basic matrix was derived. The polarization degree of reflected radiation was calculated [157].

For the Lorentz absorption profile the polarization characteristics of radiation in a line emergent from a semi-infinite medium were calculated. It was noted that the polarization degree as a function of the absorption profile value (rather than the frequency) depends only slightly on the type of this profile (Lorentz, Doppler or rectangular) [158] (with V.V. Ivanov).

Review [159].

7.7 V.P. Grinin

The non-stationary radiation fields in a semi-infinite medium with anisotropic scattering and illuminated by a parallel external flux or a point source were studied [160]. The full radiation and the dependence of the radiation density on the distance from the source were found. The solutions were expressed through the function introduced by I.N. Minin [83].

The methods to calculate the radiation fields in expanding media were proposed [161].

The concept of non-local (large-scale) radiative interaction was introduced and the equations describing the interaction were deduced [162] (coauthor S.I. Grachev) and [163].

The radiation pressure in moving media with axial symmetry was studied [164].

Reviews [165, 166].

7.8 H. Domke

With V.V. Ivanov [104].

The radiative transfer theory in spectral lines was expanded to the presence of a weak magnetic field in [167].

The problem of conservative Rayleigh scattering of polarized radiation in a semi-infinite medium was reduced to searching for several source functions depending only on the optical depth and determined by the integral equations. For the Milne and the reflection problems the number of these functions is equal to two [168]. The results were transferred to a finite layer.

The general scattering matrix was expanded to the generalized spherical functions. The corresponding radiation fields were separated into the azimuthal harmonics [169].

The singular solutions to the equation of the polarized radiation transfer were determinated and the solutions to the multiple scattering problems were expanded in elementary modes [170, 171].

The methods to calculate the transfer of polarized radiation were proposed: the doubling method [172], the application of the invariance principles [173] and of the transfer equation in the Q-form [138] expanded to polarized radiation [174] (coauthor E.G. Yanovitskij).

The transformation of the equation for H-function in order to accelerate the convergence of iterations [175], which was followed by the albedo shifting method.

Book [176].

7.9 S.I. Grachev

With V.V. Ivanov [109] and with V.P. Grinin [162].

The characteristic lengths of radiation transfer in a one-dimensional infinite medium expanding with a constant velocity gradient (the thermalization length, the thickness of the boundary layer, the diffusion length) were determinated. The asymptotic behavior of the solutions were obtained by the factoring method for the rectangular, the Doppler and the power absorption profiles [177].

The asymptotics for the resolvent functions and the source functions for particular source distributions (uniform, exponential, point source) with scattering in an infinite medium isotropically expanding with a constant (small) velocity gradient were deduced. The scattering is considered to be conservative with the Doppler or power absorption profiles [178].

The asymptotic self-similar representations of the kernel and resolvent functions, which characterize the radiation fields in a three-dimensional infinite and semi-infinite media expanding with the velocity gradient were obtained [179].

The explicit expression of the resolvent function was deduced for the problem of the non-stationary line radiation field in a semi-infinite medium for scattering with CFR and $t_2 = 0$ [180] in terms of the eigenfunctions of the basic integral equation (1) found in [146].

The polarization in a spectral line was investigated. In [181] it was shown that the asymptotic expansion for the matrix of the source functions in the problem of the line scattering with CFR and the Doppler profile could be obtained directly from the matrix equation defining it. Some particular cases of the true absorption and the depolarization values were considered. In [182] the problem of calculating the line radiation fields in the medium with uniform distribution of sources was reduced to two nonlinear equations for the matrices of the dimension 6×6. For scattering with CFR (even with the Hanle effect) the two equations were replaced by one. In [183] the asymptotic and numerical solutions to this equation were obtained. Finally, in [184] the Hanle matrix was factorized and the matrix generalization of the so-called $\sqrt{\varepsilon}$ law ($\varepsilon = 1 - \lambda$) was deduced.

Review [185].

8 Conclusion

Thus, V.V. Sobolev and his disciples have succeeded in building the analytical radiative transfer theory for monochromatic scattering as well as for scattering in spectral lines, including the scattering of polarized and non-stationary radiation. Their calculations demonstrated characteristic features of various types of scattering and are in qualitative agreement with the observational data for a variety of astrophysical objects.

Certainly several other groups have been studying the same problems. Their works are described in the reviews mentioned in the text. These groups exchanged the information and results as well as cited the works by each other. In this review we summarize only the main works by V.V. Sobolev and his school.

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