Inhomogeneous Semi-Infinite Atmospheres – On Transforming Conservative Multiple Scattering to Non-Conservative Multiple Pseudo-Scattering

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The F- and K-integrals are used to transform the zeroth azimuthal Fourier component of the radiative transfer equation for conservative multiple scattering of polarized light in vertically inhomogeneous plane atmospheres into an equivalent transfer equation with a modified phase matrix corresponding to non-conservative pseudo-scattering. As an example, the transformation to non-conservative multiple pseudo-scattering is applied to express the surface Green's function matrix for conservative scattering in terms of the surface Green's function matrix for non-conservative pseudo-scattering.

1 Introduction

The exclusive property of the transfer equation for conservative multiple scattering, which permits to determine the first and second angular moments of the intensity of the radiation field, the so called F- and K-integrals, a priori, up to two constant parameters, has been pointed out by Chandrasekhar [1] as well as by Sobolev [2] and partly employed by them on treating radiative transfer problems in vertically homogeneous conservative plane media. Here, it is shown, that even for vertically inhomogeneous conservative media, the F- and K-integrals allow us to transform the conservative radiative transfer equation into an equivalent transfer equation of the same form corresponding to non-conservative pseudo-scattering.

2 The transfer equation

Let us consider the transfer of polarized radiation in a vertically inhomogeneous and source-free plane atmosphere with local conservative scattering properties assumed to be macroscopically isotropic and mirror symmetric. It is well known (c.f. [3]) that, after azimuthal Fourier decomposition, the only conservative

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transfer equation emerges for the two-component vector of the azimuthally averaged Stokes parameters I and Q

$$u\frac{\partial}{\partial\tau}\mathbf{I}(\tau,u) = -\mathbf{I}(\tau,u) + \frac{1}{2}\int_{-1}^{+1} dv\mathbf{W}_{IQ}(\tau;u,v)\,\mathbf{I}(\tau,v), \tag{1}$$

where $\tilde{\mathbf{I}}(\tau,u) = (I(\tau,u), \quad Q(\tau,u))$. Here, the tilde denotes transposition of the vector, τ is the optical depth in the atmosphere, and u is the cosine of the polar angle with respect to the inner normal at the top $\tau = 0$ of the atmosphere. The matrix $\mathbf{W}_{IQ}(\tau;u,v)$ is the azimuthally averaged I,Q-component of the complete phase matrix. Local macroscopic mirror symmetry and reciprocity imply [3]

$$\mathbf{W}_{IQ}(\tau;u,v) = \mathbf{W}_{IQ}(\tau;-u,-v) = \tilde{\mathbf{W}}_{IQ}(\tau;v,u), \tag{2}$$

respectively. For conservative scattering, there hold the integral relations

$$\frac{1}{2}\int_{-1}^{+1} dv\,\mathbf{W}_{IQ}(\tau;u,v)\,\mathbf{i}_0 = \mathbf{i}_0, \quad \frac{1}{2}\int_{-1}^{+1} dv\,\mathbf{W}_{IQ}(\tau;u,v)v\,\mathbf{i}_0 = \frac{u}{3}\beta_1(\tau)\,\mathbf{i}_0, \tag{3}$$

where $\tilde{\mathbf{i}}_0 = (1, \quad 0)$. By means of Eq. (1) in conjunction with Eqs. (2), and (3), we find that the flux of radiative energy will be constant, i.e.,

$$F(\tau) = \frac{1}{2}\int_{-1}^{+1} du\,u\,\tilde{\mathbf{i}}_0\,\mathbf{I}(\tau,u) = F = const, \tag{4}$$

and the K-integral is found to be

$$K(\tau) = \frac{1}{2}\int_{-1}^{+1} du\,u^2\,\tilde{\mathbf{i}}_0\mathbf{I}(\tau,u) = K(0) - \left(1 - \frac{\bar{\beta}_1(\tau)}{3}\right)\tau F. \tag{5}$$

Here, $\bar{\beta}_1(\tau)$ is defined as $\bar{\beta}_1(\tau) = \frac{1}{\tau}\int_0^\tau dt\bar{\beta}_1(t)$. Finally, two eigensolutions to the transfer equation (1) can be found

$$\mathbf{i}_0(\tau,u) = \mathbf{i}_0, \quad \mathbf{i}_1(\tau,u) = \left[\left(1 - \frac{\bar{\beta}_1(\tau)}{3}\right)\tau - u\right]\mathbf{i}_0. \tag{6}$$

3 The equivalent transfer equation

On defining a modified phase matrix

$$\mathbf{W}_c(\tau;u,v) = \mathbf{W}_{IQ}(\tau;u,v) - \left[c_1(\tau)\,u^2\,\mathbf{i}_0\,\tilde{\mathbf{i}}_0\,v^2 + c_2(\tau)\,u\,\mathbf{i}_0\,\tilde{\mathbf{i}}_0\,v\right], \tag{7}$$

and replacing the phase matrix in Eq. (1) by means of Eq. (7), and using also Eqs. (4) and (5), we rewrite the conservative transfer equation (1) in the form

$$\begin{aligned} u\frac{\partial}{\partial\tau}\mathbf{I}(\tau,u) &= -\mathbf{I}(\tau,u) + \frac{1}{2}\int_{-1}^{+1} dv\,\mathbf{W}_c(\tau;u,v)\,\mathbf{I}(\tau,v) \\ &+ c_1(\tau)\,u^2\,\mathbf{i}_0\left[K(0) - \left(1 - \frac{\bar{\beta}_1(\tau)}{3}\right)\tau F\right] + c_2(\tau)\,u\,\mathbf{i}_0 F. \end{aligned} \tag{8}$$

Obviously, the new transfer equation (8) describes non-conservative multiple pseudo-scattering, with some primary (pseudo-) source terms on the r.h.s. linearly dependent on two constants F and $K(0)$, which can be determined a posteriori. We note that a particular solution to the transfer equation (8) can be found in terms of the eigensolutions (6) of the original conservative transfer equation (1)

$$\mathbf{I}_p(\tau, u) = 3\left[\mathbf{i}_0 K(0) - \mathbf{i}_1(\tau, u) F\right]. \tag{9}$$

4 Semi-infinite medium surface Green's function matrix

The semi-infinite medium surface Green's function matrix $\mathbf{G}(\tau, u; 0, \mu_0)$, with $-1 \leq u \leq +1$, $0 \leq \mu_0 \leq 1$, and $0 < \tau < \infty$, is defined as the finite solution to the transfer equation

$$u \frac{\partial}{\partial \tau} \mathbf{G}(\tau, u; 0, \mu_0) = -\mathbf{G}(\tau, u; 0, \mu_0) + \frac{1}{2} \int_{-1}^{+1} dv\, \mathbf{W}(\tau; u, v)\, \mathbf{G}(\tau, v; 0, \mu_0), \tag{10}$$

subject to the half-range boundary condition

$$\mathbf{G}(+0, \mu; 0, \mu_0) = \frac{1}{\mu} \delta(\mu - \mu_0)\, \mathbf{E}, \quad \mu, \mu_0 \in [0, 1], \tag{11}$$

at the top, where $\mathbf{E} = \mathbf{diag}(1, 1)$. In terms of the surface Green's function, the matrix of diffuse reflection is given by

$$\mathbf{R}(\mu, \mu_0) = \frac{1}{2} \mathbf{G}(+0, -\mu; 0, \mu_0), \quad \mu, \mu_0 \in [0, 1], \tag{12}$$

where μ_o denotes the direction of incidence. Reciprocity implies $\mathbf{R}(\mu, \mu_0) = \tilde{\mathbf{R}}(\mu_0, \mu)$ (c.f. [3]). There is no net flux of radiative energy for finite radiation fields in a semi-infinite conservatively scattering atmosphere without internal primary sources. Thus, the F-integral of the corresponding surface Green's function matrix $\mathbf{G}_{IQ}(\tau, u; 0, \mu_0)$ becomes zero,

$$\tilde{\mathbf{F}}_{IQ}(\tau; 0, \mu_0) = \tilde{\mathbf{F}}_{IQ}(\tau; 0, \mu_0) = \frac{1}{2} \int_{-1}^{+1} du\, u\, \tilde{\mathbf{i}}_0\, \mathbf{G}_{IQ}(\tau, u; 0, \mu_0) = \mathbf{0}. \tag{13}$$

Instead of seeking the surface Green's function matrix $\mathbf{G}_{IQ}(\tau, u; 0, \mu_0)$ as the solution to the conservative transfer equation (10) with $\mathbf{W}(\tau; u, v) = \mathbf{W}_{IQ}(\tau; u, v)$, we apply the equivalent transfer equation (8) corresponding to non-conservative pseudo-scattering, where $\mathbf{I}(\tau, u)$ is replaced by the function matrix $\mathbf{G}_{IQ}(\tau, u; 0, \mu_0)$, while $F = 0$, and $K(0)$ is replaced by the transposed vector

$$\tilde{\mathbf{K}}_{IQ}(+0; 0, \mu_0) = \frac{1}{2} \int_{-1}^{+1} du\, u^2\, \tilde{\mathbf{i}}_0\, \mathbf{G}_{IQ}(+0, u; 0, \mu_0). \tag{14}$$

On taking into account the particular solution (9), we use the surface Green's function matrix $\mathbf{G}_c(\tau, u; 0, \mu_0)$ for non-conservative pseudo-scattering to get, after some algebra, the surface Green's function matrix for conservative scattering as

$$\mathbf{G}_{IQ}(\tau, u; 0, \mu_0) = \mathbf{G}_c(\tau, u; 0, \mu_0) + \frac{3}{D}\left[\mathbf{i}_0 - \int_0^1 d\eta\, \mathbf{G}_c(\tau, u; 0, \eta)\eta\, \mathbf{i}_0\right]\tilde{\mathbf{K}}_c(+0; 0, \mu_0) \tag{15}$$

with

$$\tilde{\mathbf{K}}_c(+0; 0, \mu_0) = \frac{1}{2}\left[\mu_0\, \tilde{\mathbf{i}}_0 + 2\int_0^1 d\mu\, \mu^2\, \tilde{\mathbf{i}}_0\, \mathbf{R}_c(\mu, \mu_0)\right], \tag{16}$$

and $D = 3\int_0^1 d\eta\, \tilde{\mathbf{K}}_c(+0; 0, \eta)\, \eta\, \mathbf{i}_0$, while $\tilde{\mathbf{K}}_c(+0; 0, \mu_0) = D\, \tilde{\mathbf{K}}_{IQ}(+0; 0, \mu_0)$. It is easy to verify that $\mathbf{G}_{IQ}(\tau, u; 0, \mu_0)$ as given by Eq. (15) satisfies the correct transfer equation (8) as well as the boundary condition (11). When specified with $\tau = +0$ and $u = -\mu$, Eq. (15) provides a simple formula for retrieving the reflection matrix $\mathbf{R}_{IQ}(\mu, \mu_0)$ for conservative scattering by means of the reflection matrix $\mathbf{R}_c(\mu, \mu_0)$ for non-conservative pseudo-scattering

$$\mathbf{R}_{IQ}(\mu, \mu_0) = \mathbf{R}_c(\mu, \mu_0) + \frac{3\,(1 - D)}{D^2\gamma}\, \mathbf{K}_c(+0; 0, \mu)\, \tilde{\mathbf{K}}_c(+0; 0, \mu_0), \tag{17}$$

where the constant $\gamma = 3\int_0^1 d\eta\, \eta^2\, \tilde{\mathbf{i}}_0\, \mathbf{K}_{IQ}(+0; 0, \eta) = \frac{3}{D}\int_0^1 d\eta\, \eta^2\, \tilde{\mathbf{i}}_0\, \mathbf{K}_c(+0; 0, \eta)$ is the so called extrapolation length well known in radiative transfer theory.

For practical methods to calculate reflection matrices for inhomogeneous semi-infinite atmospheres, which are applicable also to compute $\mathbf{R}_c(\mu, \mu_0)$ for non-conservative pseudo-scattering, we refer to the textbook of Yanovitsky [4] and references therein. Finally, we note that for homogeneous atmospheres the transformation to equivalent pseudo-scattering with reduced effective albedo of single scattering can be performed also for non-conservative scattering. This has been described in an earlier paper [5].

References

1. *S. Chandrasekhar*, Radiative Transfer. New York: Oxford University Press, 1950.

2. *V. V. Sobolev*, Light Scattering in the Atmospheres of Planets. Moscow: Nauka, 1972 (in Russian). Translated as Light Scattering in Planetary Atmospheres. Oxford: Pergamon Press, 1975.

3. *J. W. Hovenier, C. van der Mee, H. Domke*, Transfer of Polarized Light in Planetary Atmospheres: Basic Concepts and Practical Methods. Amsterdam: Elsevier, 2005.

4. *E. G. Yanovitsky*, Light Scattering in Inhomogeneous Atmospheres. New York: Springer Verlag, 1997.

5. *H. Domke*, Eigenvalue shifting – a new analytical-computational method in radiative transfer theory. In Photopolarimetry in Remote Sensing. Eds. G. Videen et al. Dordrecht: Kluwer Academic Publ., 2004, pp. 107–124.