The Sobolev Approximation in the Development and Astrophysical Applications

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The Sobolev approximation is one of the most effective methods of the modeling of emission spectra of astrophysical objects of various types. It plays also an important role in the radiative hydrodynamics. In this short review, after an introduction to the Sobolev method, I discuss the main steps in its development and astrophysical applications.

1 Introduction to the Sobolev method

Emission spectra of many astrophysical objects are formed in the media with large-scale differential motions which velocities are much greater than the thermal velocity of atoms. In these conditions, the Doppler shift of the radiation frequency leads to strong changes in optical properties of the gas in the line frequencies. This circumstance strongly complicates the solution of the radiative transfer problem. However, as shown by V.V. Sobolev [58], in the media with the large velocity gradient, the solution of this problem can be significantly simplified.

The essence of this approximation is as follows: for large velocity gradients, due to the shift between resonance frequencies of the emitting and absorbing atoms, the radiative interaction at each point of the medium $\vec{r}$ is determined by its local vicinity. The characteristic size of this vicinity is equal to the distance from the given point to that, where the aforementioned shift in resonance frequencies is equal to the half-width of the line profile function $\Delta\nu_D$ determined by the thermal (or turbulent) velocity v_t,

$$s_0 = v_t/|dv_{\vec{s}}/ds|. \tag{1}$$

Here $dv_{\vec{s}}/ds$ is the velocity gradient in the comoving coordinate system at the point $\vec{r}$ in the direction $\vec{s} = \vec{r}\,' - \vec{r}$; $s = |\vec{s}\,| \ll |\vec{r}\,|$), and $dv_{\vec{s}}/ds \simeq [\vec{v}(\vec{r}\,') - \vec{v}(\vec{r})]/s$ (see Fig. 1).

For rough estimates, the velocity gradient in this expression can be replaced by the ratio v/R, where v is the characteristic velocity of large-scale motions and R is the characteristic size occupied by the emitting gas. As a result, we obtain the approximate relation: $s_0 \approx R(v_t/v)$. The parameter s_0, which was subsequently called the “Sobolev length”, is the main parameter of the Sobolev method, characterizing the size of the local vicinity of the point.

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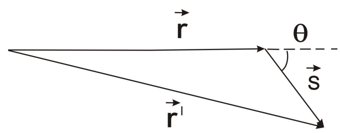

Figure 1

In general case, the equation for the source function has a form

$$S(\vec{r}) = \lambda \int_V K(\vec{r}, \vec{r}\,')\, S(\vec{r}\,')\, d\vec{r}\,' \, + \, g(\vec{r}). \tag{2}$$

Here $K(\vec{r}, \vec{r}\,')$ is the kernel function determining the density probability of a transfer of the radiative excitation from the point $\vec{r}$ to the point $\vec{r}\,'$, λ is the probability of a photon survival at a single scattering, V is the volume of the space filled in with atoms, and $g(\vec{r})$ represents the primary sources of excitation in the spectral line under consideration.

In the media with the large velocity gradient $s_0 \ll R$, Eq. (2) can be essentially simplified. In this case, one can approximately assume that the source function does not change in the vicinity of the point $\vec{r}$, and we can take it outside the integral by setting $S(\vec{r}\,') \approx S(\vec{r})$. A similar procedure can be done with the kernel function $K(\vec{r}, \vec{r}\,')$ if to replace its parameters that determine optical properties of the medium (atomic level populations, thermal or turbulent velocity) by the corresponding values at the point $\vec{r}$. Finally, one can neglect an influence of the boundaries and assume that the medium fills in an infinite volume of space. As a result, the integral equation with the very complicated kernel is transformed to the simple equation

$$S(\vec{r})\ [1 - \lambda + \lambda\beta(\vec{r})] = g(\vec{r}), \tag{3}$$

where β is the probability of a photon to escape the point of the medium $\vec{r}$ without scattering and absorption along the way

$$\beta(\vec{r}) = 1 - \int K(\vec{r}, \vec{r}\,')\, d\vec{r}\,'. \tag{4}$$

It should be noted that in the stationary medium the corresponding kernel function is always normalized to unit. This reflects the obvious fact that a photon emitted in an infinite medium will be absorbed somewhere in it. A principal difference of the radiative diffusion in a medium with a velocity gradient is that this normalization condition is violated, and the integral of the kernel function over infinite space is always less than unity. This means that because of enlightenment of the medium in the line frequencies due to the Doppler effect, there is a nonzero probability for a photon to escape from the point of the medium lying formally at the infinite distance from its boundary: $\beta(\infty) > 0$. This property of the radiation transfer in the line frequencies in moving media is the basis of the Sobolev approximation (SA).

It is important that the photon escape probability from an arbitrary point of the medium is expressed fairly simply in terms of the characteristics of the medium and the velocity field at the given point. For example, in a spherically-symmetric envelope expanding with the velocity $v(r)$ we have

$$\beta(r) = \int_0^1 \frac{1 - e^{-\tau(r,\mu)}}{\tau(r,\mu)}\, d\mu, \tag{5}$$

where $\tau(r,\mu)$ is the effective optical depth of the medium at the point r in the direction $\vec{s}$ forming an angle $\theta = \arccos\mu$ with the vector $\vec{r}$

$$\tau(r,\mu) = k(r)\, v_t |\psi(r,\mu)|^{-1}, \tag{6}$$

$k(r)$ is the integrated line opacity per unit volume (weighted with the line profile function), and

$$\psi(r,\mu) = \frac{dv_{\vec{s}}}{ds} = \frac{dv}{dr}\,\mu^2 + \frac{v}{r}\,(1-\mu^2). \tag{7}$$

In the particular case of an isotropically expanding medium (an example of which is the expanding Universe), $v(r) = Ar$, $\psi(r,\mu) = v/r$ and $\tau(r,\mu) = \mathit{constant} = k\, v_t/A$. As a result, $\beta(r) = (1-e^{-\tau(r)})/\tau(r)$. From this, we get $\beta = 1/\tau$ for $\tau \gg 1$.

Thus, if the primary sources of excitation $g(\vec{r})$ in the spectral line are known, then we can immediately calculate the source function from Eq. (3), and then calculate the intensity of the spectral line. This method has been originally developed by V.V. Sobolev for the case of a rectangular line profile function and the complete frequency redistribution in a comoving frame. Later, in 1957, he considered in [59] the general case of an arbitrary absorption coefficient. It turned out that the expression for the photon escape probability $\beta(r)$ does not depend on the type of the line profile function. This invariance is one of the most interesting properties of the process of radiative diffusion at line frequencies in moving media which has no analog in the case of stationary media. In the latter case, as we know, the escape probability depends sensitively on the type of the line profile function (see, e.g., the book of Ivanov [42]).

Due to its simplicity, the Sobolev approximation was widely used when modeling and interpreting the emission spectra of stars with circumstellar envelopes and other astrophysical objects. The role of this method in the solution of complex, multilevel problems was especially great. First steps in this direction were taken by Rublev [54, 55], Gorbatskii [20], Boyarchuk [4], Doazan [11], Luud and Il'mas [45], Gershberg and Shnol [18], Grinin and Katysheva [28], Castor and Lamers [7], Natta and Giovanardi [50], and many others (see a more detailed bibliography of the works on this subject in the review [24]).

In 1961 Sobolev's book "Moving Envelopes of Stars" was translated into English by S. Payne-Gaposchkin and published in the USA. Soon a series of fundamental discoveries in astronomy have been done, resulting in appearance of new astrophysical objects: quasars, neutron stars and maser sources. First observations of ultraviolet spectra of stars from space led to the discovery of intense mass outflows (stellar winds) from hot supergiants. All this expanded considerably the field of the application of the SA and stimulated its further development in the papers by Castor [5, 6], Grachev [22], Rybicki and Hummer [57], Hummer and Rybicki [36, 37], Jeffery [39], Hutsemekers and Surdej [38], Petrenz and Puls [53], Dorodnitsyn [13], Grinin and Tambovtseva [30], and others (see below). In particular, the Sobolev approximation has been adapted for studying polarization in spectral lines [39], for the case of relativistic motions [38, 40] and conditions near black holes and neutron stars [12, 13, 14].

1.1 The SEI algorithm

The questions of the accuracy of the SA and the limits of its applicability naturally emerged. The development of numerical and asymptotic methods of the radiative transfer theory has made possible the solution of this problem. It turned out that the limits of applicability of the SA depend sensitively on the type of the line profile function and are determined by the asymptotic behavior of the kernel function in Eq. (2). These and related topics are discussed in more detail in the review papers by Grachev [23] and Grinin [24].

Using the numerical methods, Bastian et al. [2] and Hamann [31] investigated in detail the accuracy of the SA in models of spherically symmetric outflows. They have shown that the error in the calculations performed on the basis of the SA arises mainly when calculating the line profile, while accuracy of the source function calculations was quite good. Based on this result, Bertout [3] and Lamers et al. [44] suggested to use the exact expression for the intensity of the radiation emerging from the medium in the combination with the source function calculated in the Sobolev approximation. This algorithm is known as the Sobolev Exact Integration (SEI) method [44]. It yields a considerable gain in accuracy of the line profiles and is widely used when modeling the emission spectra.

2 The non-local approximation

In the seventies it was found that the presence of the large gradient velocity in the emitting region actually still does not guarantee the condition of locality of the radiative interaction at the line frequencies, and one additional condition must be fulfilled. Namely, the derivative of the velocity in the comoving coordinate system, $dv_{\vec{s}}/ds$ has to be a positive definite function of the angle θ between the vector $\vec{r}$ and the arbitrary direction $\vec{s}$. In the case of radially symmetric motions, this condition is fulfilled for the outflow with the acceleration $(dv/dr > 0)$ and does not fulfilled in the case of decelerated outflows. The latter is also true in the case of accretion flows, including the quasi-Keplerian disks. In these cases, at each point $\vec{r}$ there are directions along which $dv_{\vec{s}}/ds = 0$, and the Sobolev length $s_0 = \infty$ (see Fig. 2).

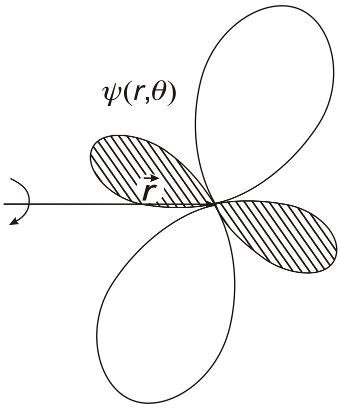

Figure 2: Azimuthal structure of the velocity gradient in the comoving frame in the plane of the quasi-Keplerian disk (from [29]). In the filled regions $\psi(r,\theta) < 0$.

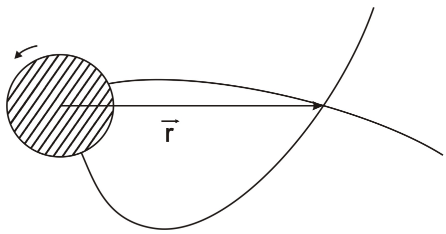

Figure 3: The common-point velocity surface in the rotating and collapsing envelope (from [24]).

These distinctions in the structure of the velocity field in the comoving frame are key for definition of a type of the radiative interaction in a moving medium. The equation for the source function in the case of the non-local radiative interaction was first obtained by Grachev and Grinin [21]. They showed that in shells expanding with deceleration, the source function is determined by the equation

$$S(\vec{r})\;[1-\lambda+\lambda\beta(\vec{r})] = \lambda\int_{\Omega_c} S(\vec{r}\,')\;\left[1-e^{-\tau(\vec{r}\,',\theta')}\right]\,\beta(\vec{r},\theta)\,\frac{d\Omega}{4\pi}+g(\vec{r}), \tag{8}$$

which differs from Eq. (3) by the integral term. This additional term allows for the fact that besides the local vicinity of the point, a contribution to photoexcitations at the point $\vec{r}$ comes from the so-called surfaces of comoving points that are in resonance with the point $\vec{r}$ and satisfy the equation $(\vec{v}(\vec{r}\,')-\vec{v}(\vec{r}))\cdot\vec{s}=0$. An example of such a surface is presented in Fig. 3. The existence of such surfaces provides for the non-local nature of the radiative interaction in media moving with large velocity gradients.

An equation similar to Eq. (8) was derived independently and investigated in detail by Rybicki and Hummer [56]. They introduced the term "the common points surface" which has now become generally accepted. We should also note a paper by Deguchi and Fukui [9] in which Eq. (8) was derived and used for calculations of the spectra of collapsing protostellar clouds as well as a version of a non-local radiative interaction between components of a resonance doublet in moving media, considered by Surdej [60].

Let us note that the integral term in this equation does not contain any new quantities in comparison with the local version of the Sobolev approximation. We see the same expressions for the escape probability β and for the effective optical depth τ as in Eq. (3). An analysis shows that the contribution of the integral term in Eq. (8) to the source function depends on two factors: on the behavior of the primary sources of excitation $g(r)$ and on the solid angle Ω_c in which the surface of comoving points is seen from the point $\vec{r}$. For $\Omega_c \ll 4\pi$, the influence of the integral term on the source function can be neglected in most cases. The non-local version of the Sobolev approximation is applied to modeling the emitting regions around young stars (see, e.g., the works of Hartmann et al. [33], Muzerolle et al. [49]), black holes and neutron stars (the paper of Dorodnitsyn [14]), supernova shells (that of Fransson [16]) and other astrophysical objects. It is interesting to note that the non-local radiative interaction can take place near to the compact objects (black holes and neutron stars) in the *accelerating outflows* due to the gravitation red shift (the work of Dorodnitsyn [13]). In this case the P Cygni profile may have both red- and blue-shifted absorption troughs (in contrast with the classical theory).

3 The radiative force

Due to the large cross sections of the interaction with matter, radiation in spectral lines plays an important role in the dynamics of gas in high-luminosity astrophysical objects. In 1957 Sobolev [59] has obtained the formula for the radiative force in the plane-parallel layer expanding with a constant velocity gradient. Only diffuse radiation produced in the layer was taken into account. The next important step was made by Castor [5]. Developing the ideas laid down in the SA, he has obtained the expression for the radiative force exerted on the gas in a spherically symmetric expanding shell with absorption and scattering of continuous stellar radiation in the line frequencies. It has a simple form

$$f_{r,L} = \frac{k(r)\, F_c\, \Delta\nu_D}{c}\, \min(1, 1/\tau), \tag{9}$$

where F_c is the radiation flux at the stellar surface at the frequency of the spectral line under consideration, k is the integrated line opacity (normalized to a unit of mass), c is the speed of light, and τ is the optical depth defined by Eqs. (6)–(7). Castor, Abbott, and Klein (CAK) [8] subsequently calculated models of the expanding envelopes of hot supergiants and showed that the main contribution to the radiative force comes from a set of weak subordinate lines of ionized atoms such as CII, CIII, etc. It is needed to note that in the earlier attempts to solve this problem it was assumed that the main contribution to the radiative force provided the ultraviolet resonance lines (see the works of Lucy and Solomon [46], Lucy [47]). However, their effect was too small to explain the high mass loss rate from the hot supergiants.

The CAK theory has had a significant effect on the development of the theory of radiative driven stellar winds. Due to the efforts of Castor and his co-authors, this theory is now one of the most advanced fields of theoretical astrophysics. The results of this theory are applied not only to calculations of the radiative driven winds from hot stars but also to modeling of the envelopes and disk winds of quasars and active galactic nuclei.

3.1 The azimuthal component of the radiative force

Completing the topic of the radiative force in moving media, we note one non-trivial property of this mechanism. It consists in the fact that in envelopes with axially symmetric motions, along with the radial component $f_{r,L}$ of the radiative force there is also an azimuthal component $f_{\theta,L}$. Its appearance is related to the fact that in the general case of axially symmetric motions, the derivative of the velocity in the comoving frame in the plane of the motions includes the odd dependence on the angle θ between the vector $\vec{r}$ and the arbitrary direction $\vec{s}$ in the plane of motions

$$\frac{dv_{\vec{s}}}{ds} = \frac{dv}{dr}\cos^2\theta + \frac{v}{r}\sin^2\theta + \left(\frac{du}{dr} - \frac{u}{r}\right)\sin\theta\,\cos\theta. \tag{10}$$

For this reason, the Sobolev length $s_0(r,\theta)$ and, therefore, the optical properties of the medium at the line frequencies are asymmetric functions of the angle θ (Fig. 2). Radiation in the spectral line propagates in such a medium not along the radius vector $\vec{r}$ but at some angle to it. This angle depends on the ratio between the radial and tangential velocity components v and u. A result of this is an azimuthal component of the radiative force. Its sign depends on the physical conditions in the envelope, such as the gradient of the source function and the direction of the radial velocity (expansion or accretion). Depending on these parameters, the direction of the azimuthal radiative force can either coincide with the rotation of a gaseous envelope or act against the rotation. The efficiency of this mechanism, operating on the principle of "Segner's wheel", depends on the ratio between the radiation density at spectral lines frequencies and the kinetic energy of gas as well as on the ratio between the velocity components v and u. For example, in the accretion disks $v \ll u$, and the ratio $f_{\theta,L}/f_{r,L} \sim v/u \ll 1$. More detailed information on this radiative mechanism can be found in [25, 26, 27]. Here we only note that in 1995 Owocki, Cranmer, and Gayley [52] independently discovered a similar effect in a numerical solution of the radiative hydrodynamics equations in the envelopes of Be stars (see also [17]).

4 Molecular lines and cosmic masers

Despite the fact that the velocities of internal motions in interstellar molecular clouds do not exceed several kilometers per second, as a rule, they nevertheless can also be objects to which SA can be applied. Because of the low temperatures, the velocities of thermal motions of molecules in the clouds are also very small and often do not exceed several hundreds of meters per second. This fact has been used by many authors who have used the SA for the diagnostics of interstellar clouds based on molecular line intensities. In particular, Goldreich and Kylafis [19], and Deguchi and Watson [10] used the SA to study the polarization of molecular lines.

In cosmic masers the optical depth of the emitting region with an inverted population of molecular levels plays the role of the amplification factor, and in the case of powerful masers it can be much larger than unity. Under these conditions, even a relatively small number of working molecules going out of resonance can cause considerable changes in the maser emission intensity (e.g., the paper of Watson and Wyld [63]). Maser lines are therefore the most sensitive indicators of internal motions of the medium, especially in the case of unsaturated regime. For the same reason, maser emission is also very sensitive to the type of radiative interaction in the medium. Really, only at the non-local radiative interaction, at each point of the medium there are directions in which the velocity gradient in the comoving coordinate system equals zero. In particular, in the quasi-Keplerian disk the equation $\psi(r, \mu) = 0$ has four solutions: $\mu_1, \ldots, \mu_4$ (Fig. 2). In terms of the SA, the optical depth in the line frequencies in these directions formally becomes infinite.

To determine the effective optical depth in this case, one must allow for the second order derivative of the velocity (see Fig. 4) from the work of Grinin and Grigor'ev [29]). Typical examples of such objects are the quasi-Keplerian disks (see, e.g., the paper by Babkovskaia et al. [1]) as well as protostellar clouds in the phase of gravitational contraction. It should be also noted that in addition to a velocity gradient, the maser lines can be also sensitive to the presence of magnetic field. These are the hydroxyl masers and some others. Calculations of such masers require joint allowance for the velocity and magnetic field gradients (see the work of Kegel and Varshalovich [41]).

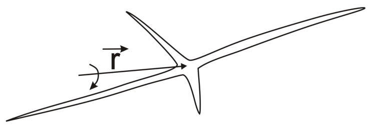

Figure 4: Azimuthal structure of the optical depth in the maser line in the plane of the quasi-Keplerian disk (from [29]).

Thus, despite the appearance of the effective numerical methods (see, e.g., [48, 35, 34] and references there), the Sobolev approximation is an important tool for modeling and diagnostics of emitting regions in the different kinds of astrophysical objects. It is applied in combination with the Monte Carlo method for modeling of the complex emitting regions near the young stars (see, e.g., the papers of Harries [32], Kurosawa et al. [43]). The unique object of application of the SA is the expanding Universe (e.g., the work of Dubrovich and Grachev [15]) which can be optically very thick for the L_α radiation at the large red shifts as shown by Varshalovich and Syunyaev [62]. It is also difficult to overestimate the role of the SA in the theory of the radiative driving stellar winds (see the paper of Owocki and Puls [51] and references therein).

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