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UCAC4: Stellar kinematics with vector spherical functions

V.V. Vityazev⋆ and **A.S. Tsvetkov**

Sobolev Astronomical Institute, Saint-Petersburg University, Bibliotechnaya pl. 2, St. Petersburg, 198504 Russia

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We present a method of kinematic analysis of proper motions by vector spherical functions, and the results of its application to astrometric data. The sets of vector spherical functions which are orthonormal on a full sphere as well as on a latitude zone are constructed. Decomposition of the proper motions into a set of such functions allows model-independent study of stellar kinematics. If needed, the parameters of the standard (say, Ogorodnikov-Milne) model may be derived from the coefficients of the decomposition. In contrast to the commonly used least squares estimation of the model's parameters, vector spherical functions identify all systematic components of the velocity field (no matter, whether they are incorporated into the model or not) and give us a possibility to test whether the data are compatible with the model. In this paper, we apply this technique for the first time to the proper motions from the UCAC4 catalog for stars in the 11 to 16 magnitude range. We derive all-sky solutions and the solutions based on stars in the northern and southern Galactic hemispheres. The all-sky solution provide evidence for noticeable magnitude-dependent trends in the coordinates of the solar motion apex, Oort constants, angular speed of the local Galactic rotation, and the slope of the local rotation velocity curve as we go from bright to faint stars. Furthermore, our all-sky vector spherical function analysis identified strong and reliable extra-model harmonics, whereas the solutions for the northern and southern hemisphere indicate sign reversals for some of the Ogorodnikov-Milne parameters. We show that both effects appear simultaneously and can be explained by the slowdown of Galactic rotation with increasing distance from the main Galactic plane. We estimate the absolute value of the vertical gradient of the Galactic rotational velocity to be $\sim 40\ \mathrm{km\,s^{-1}\,kpc^{-1}}$.

1 Introduction

During recent 15 years we were witnessing a real parade of catalogues implementing the ICRS in optical waves with full coverage of the sky. The first one was the Hipparcos ($\sim$100 thousand stars) observed in space and tied to the ICRF-1 (Perryman et al. 1997). It was followed by Tycho-2 (~ 2.5 million stars) created from ground based and space-based observations (Høg et al. 2000). The next came UCAC3 (Zacharias et al. 2010) which was soon replaced by UCAC4 (Zacharias et al. 2013). The UCAC4 is an all-sky catalogue containing about 113 million stars covering mainly the 8 to 16 magnitude range in a single bandpass between V and R. The positional accuracy of stars in UCAC4 at mean epoch is about 15–100 mas per coordinate, depending on magnitude, while the formal errors in PMs range from about 1 to 10 mas yr^{-1} depending on magnitude and observing history. Systematic errors in PMs are estimated to be about 1–4 mas yr^{-1}. All bright stars have been added to UCAC4 from Hipparcos and Tycho-2 catalogues. The UCAC4 may be considered complete to about $R = 16$. At present, the largest catalogue of positions and proper motions is PPMXL (Roeser et al. 2010). It contains about 900 million objects and probably is full from the brightest stars down to about magnitude $V = 20$ full-sky with absolute proper motions in the ICRS reference frame. The mean errors of the proper motions range from 4 mas yr^{-1} to more than 10 mas yr^{-1}. The accuracy of positions are estimated to be 80–120 mas (at epoch 2000.0).

Modern astrometric catalogues provide a qualitatively new material, in particular, for investigating the kinematics of nearby stars. Highly accurate measurements of parallaxes, proper motions, and radial velocities for hundreds of millions of stars planned in the future space project GAIA are a motivation for developing new methods of kinematic analysis of stars. The papers by Vityazev & Shuksto (2005) and Vityazev & Tsvetkov (2009, 2011, 2012), which are devoted to the application of vector spherical harmonics (VSH) to problems of stellar kinematics, meet this requirement.

The VSH formalism is particularly well suited for analyzing the present and future catalogues containing all three components of the velocity vector – the proper motions in both coordinates and the radial velocity. The method of VSH is capable of revealing all systematic components in the stellar velocity field without resorting to a specific physical model. Comparison of the decomposition coefficients for a particular kinematic model with the observational data can reveal systematic components that are not described by the model considered. The VSH method was successfully applied to stellar kinematics by Makarov & Murphy (2007) and Bobylev et. al. (2011). Note that two-dimensional vector spherical functions were first used by

⋆ Corresponding author: vityazev@list.ru

Mignard & Morando (1990) in astrometric problems related to the comparison of catalogues to represent the systematic differences between Hipparcos and FK5. Further extensive study of this technique aiming at its application in the GAIA project may be found in recent paper by Mignard & Klioner (2012).

This paper is devoted to application of the vector spherical function formalism for kinematic analysis of proper motions of the UCAC4 catalogue. For this purpose we introduce the system of vector spherical functions orthonormal on the whole sphere as well as a system of zone vector functions orthonormal on a zone of Galactic latitudes (declinations). Next, we show that the zone spherical harmonics can be used to estimate the parameters using at least two – the main and alternative – techniques. A comparison of the main and alternative solutions allows the standard kinematic model to be tested for compatibility with the observational data. These systems of functions are used for the all-sphere analysis and for the half sphere analysis in the northern and southern Galactic hemispheres of the UCAC4 catalogue.

2 Ogorodnikov-Milne equations

The equations of the Ogorodnikov-Milne model (Ogorodnikov 1965) are commonly used to investigate stellar kinematics. In this model, the star's velocity $\boldsymbol{V}$ relative to the Cartesian Galactic coordinate system with unit vectors $\boldsymbol{e}_X, \boldsymbol{e}_Y, \boldsymbol{e}_Z$ is given by the following linear equations:

$$\boldsymbol{V} = -U\,\boldsymbol{e}_X - V\,\boldsymbol{e}_Y - W\,\boldsymbol{e}_Z + M^+\boldsymbol{r} + M^-\boldsymbol{r}, \tag{1}$$

where

$$M^+ = \begin{bmatrix} M^+_{11} & M^+_{12} & M^+_{13} \\ M^+_{21} & M^+_{22} & M^+_{23} \\ M^+_{31} & M^+_{32} & M^+_{33} \end{bmatrix}; \tag{2}$$

$$M^- = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}. \tag{3}$$

Here $\boldsymbol{r}$ is the heliocentric vector toward a star; U, V, W are the components of the solar motion vector relative to the stellar centroid; Ω_1, Ω_2, and Ω_3 are the components of the rigid-body rotation vector of the stellar centroid; M^+_{11}, M^+_{22}, M^+_{33} are the parameters describing the contraction or expansion of the velocity field along the principal axes of the coordinate system; $M^+_{12} = M^+_{21}$, $M^+_{13} = M^+_{31}$, $M^+_{23} = M^+_{32}$ - parameters describing the velocity field deformation in the principal plane and the two planes perpendicular to it.

To connect the components of $\boldsymbol{V}$ with the radial velocity V of a star, and its proper motions μ_l and μ_b in longitude and latitude, respectively, let us project the vector $\boldsymbol{V}$ onto the unit vectors $\boldsymbol{e}_l, \boldsymbol{e}_b, \boldsymbol{e}_r$. Introducing the factor $\mathcal{K} = 4.74$ for converting the dimensions of stellar proper motions mas yr^{-1} to km s^{-1} kpc^{-1}, we obtain

$$\begin{bmatrix} \mathcal{K}\mu_l\cos b \\ \mathcal{K}\mu_b \\ V/r \end{bmatrix} = A(l,b)\begin{bmatrix} U/r \\ V/r \\ W/r \end{bmatrix} + $$

$$+\, A(l,b)\,(M^+ + M^-)\begin{bmatrix} \cos b\cos l \\ \cos b\sin l \\ \sin b \end{bmatrix}, \tag{4}$$

where A is the transformation matrix from the unit vectors $\boldsymbol{e}_X, \boldsymbol{e}_Y, \boldsymbol{e}_Z$ of the Cartesian Galactic coordinate system to the unit vectors $\boldsymbol{e}_l, \boldsymbol{e}_b, \boldsymbol{e}_r$ directed along the direction of change in Galactic longitude and latitude and the line of sight:

$$A(l,b) = \begin{bmatrix} -\sin l & \cos l & 0 \\ -\cos l\sin b & -\sin l\sin b & \cos b \\ \cos l\cos b & \sin l\cos b & \sin b \end{bmatrix}. \tag{5}$$

3 Scalar spherical functions

Spherical functions are widely used in various areas of mathematics and physics. Their definition can be found in many sources, for example in Arfken (1970). In this paper, we will use the following representation for them:

$$K_{nkp}(l,b) = R_{nk}\begin{cases} P_{n,0}(b), & k=0,\ p=1; \\ P_{nk}(b)\sin kl, & k\neq 0,\ p=0; \\ P_{nk}(b)\cos kl, & k\neq 0,\ p=1, \end{cases} \tag{6}$$

$$R_{nk} = \sqrt{\frac{2n+1}{4\pi}}\begin{cases} \sqrt{\frac{2(n-k)!}{(n+k)!}}, & k>0; \\ 1, & k=0, \end{cases} \tag{7}$$

where l and b are the longitude and latitude of the point on the sphere, respectively, $(0 \le l \le 2\pi;\ -\pi/2 \le b \le \pi/2)$; $P_{nk}(b)$ are the Legendre (at $k=0$) and associated Legendre (for $k>0$) polynomials that can be calculated using the recurrence relations

$$\begin{aligned} &P_{nk}(b) = \sin b\tfrac{2n-1}{n-k}P_{n-1,k}(b) - \tfrac{n+k-1}{n-k}P_{n-2,k}(b), \\ &{\scriptstyle k=0,\,1,\,\dots n=k+1,\,k+2,\,\dots} \\ &P_{kk}(b) = \tfrac{(2k)!}{2^k k!}\cos^k b, \\ &P_{k+1,k}(b) = \tfrac{(2k+2)!}{2^{k+1}(k+1)!}\cos^k b\sin b. \end{aligned} \tag{8}$$

For convenience, a linear numeration of the functions K_{nkp} by one index j is often introduced. In this way the index j is used instead of (nkp), where

$$j = n^2 + 2k + p - 1. \tag{9}$$

4 Vector spherical harmonics (VSH)

Consider a set of mutually orthogonal unit vectors $\boldsymbol{e}_l, \boldsymbol{e}_b, \boldsymbol{e}_r$ in the directions of the longitude and latitude and along the line of sight, respectively, in a plane tangential to the sphere. Using the definitions of VSFs in Arfken (1970) let us introduce radial, $\mathbf{V}_j$, toroidal, $\mathbf{T}_j$ and spheroidal $\mathbf{S}_j$ via the relations

$$\mathbf{V}_j(l,b) = K_j(l,b)\boldsymbol{e}_r, \tag{10}$$

$$\mathbf{T}_j = r_n\left(\frac{\partial K_j(l,b)}{\partial b}\boldsymbol{e}_l - \frac{1}{\cos b}\frac{\partial K_j(l,b)}{\partial l}\boldsymbol{e}_b\right), \tag{11}$$

$$\mathbf{S}_j = r_n \left(\frac{1}{\cos b} \frac{\partial K_j(l,b)}{\partial l} \boldsymbol{e}_l + \frac{\partial K_j(l,b)}{\partial b} \boldsymbol{e}_b \right), \tag{12}$$

where

$$r_n = \frac{1}{\sqrt{n(n+1).}} \tag{13}$$

Denote the components of the unit vector $\mathbf{e}_l$ as T_j^l S_j^l, and the components of the unit vector $\mathbf{e}_b$ – respectively T_j^b and S_j^b:

$$\mathbf{T}_j = T_j^l \boldsymbol{e}_l + T_j^b \boldsymbol{e}_b, \tag{14}$$

$$\mathbf{S}_j = S_j^l \boldsymbol{e}_l + S_j^b \boldsymbol{e}_b. \tag{15}$$

These components are defined as:

$$T_j^l = \frac{R_{nk}}{\sqrt{n(n+1)}} \begin{cases} P_{n,1}(b),_{k=0,\,p=1}, \\ (-k\tan b P_{nk}(b) \\ +P_{n,k+1}(b))\sin kl,_{k\neq0,\,p=0}, \\ (-k\tan b P_{nk}(b) \\ +P_{n,k+1}(b))\cos kl,_{k\neq0,\,p=1}; \end{cases} \tag{16}$$

$$T_j^b = \frac{R_{nk}}{\sqrt{n(n+1)}} \begin{cases} 0,_{k\neq0,\,p=1}, \\ -\frac{k}{\cos b} P_{nk}(b)\cos kl,_{k\neq0,\,p=0}, \\ +\frac{k}{\cos b} P_{nk}(b)\sin kl,_{k\neq0,\,p=1}; \end{cases} \tag{17}$$

$$S_j^b = \frac{R_{nk}}{\sqrt{n(n+1)}} \begin{cases} P_{n,1}(b),_{k=0,\,p=1}, \\ (-k\tan b P_{nk}(b) \\ +P_{n,k+1}(b))\sin kl,_{k\neq0,\,p=0}, \\ (-k\tan b P_{nk}(b) \\ +P_{n,k+1}(b))\cos kl,_{k\neq0,\,p=1}; \end{cases} \tag{18}$$

$$S_j^l = \frac{R_{nk}}{\sqrt{n(n+1)}} \begin{cases} 0,_{k=0,\,p=1}, \\ +\frac{k}{\cos b} P_{nk}(b)\cos kl,_{k\neq0,\,p=0}, \\ -\frac{k}{\cos b} P_{nk}(b)\sin kl,_{k\neq0,\,p=1}. \end{cases} \tag{19}$$

The functions introduced above satisfy the relations:

$$\int_\Omega (\mathbf{V}_i \cdot \mathbf{V}_j)\,\mathrm{d}\omega = \int_\Omega (\mathbf{T}_i \cdot \mathbf{T}_j)\,\mathrm{d}\omega = $$
$$= \int_\Omega (\mathbf{S}_i \cdot \mathbf{S}_j)\,\mathrm{d}\omega = \begin{cases} 0, & i \neq j, \\ 1, & i = j; \end{cases} \tag{20}$$

$$\int_\Omega (\mathbf{V}_i \cdot \mathbf{T}_j)\,\mathrm{d}\omega = \int_\Omega (\mathbf{V}_i \cdot \mathbf{S}_j)\,\mathrm{d}\omega = $$
$$= \int_\Omega (\mathbf{S}_i \cdot \mathbf{T}_j)\,\mathrm{d}\omega = 0,\ \forall i,j. \tag{21}$$

In other words, functions $\mathbf{V}_j$, $\mathbf{T}_j$, $\mathbf{S}_j$ form an orthonormal set of functions on the sphere.

5 Vector spherical harmonics for a zonal catalogue (ZVSF)

VSHs are specified for all points of the sphere where they are orthogonal and complete (Arkfen 1970). If the data are available within some latitude (declination) zone, a complete system of orthogonal functions can also be introduced. Let the data of some zonal catalogue belong to the following domain of the celestial sphere:

$$Z = \begin{cases} 0 \leq l \leq 2\pi, \\ b_{\min} \leq b \leq b_{\max}. \end{cases} \tag{22}$$

Let us introduce the transformation

$$\hat{b} = \arcsin(\alpha \sin b + \beta), \tag{23}$$

that for

$$\alpha = \frac{2}{s_2 - s_1}, \quad \beta = -\frac{s_2 + s_1}{s_2 - s_1}, \tag{24}$$

$$s_1 = \sin b_{\min}, \quad s_2 = \sin b_{\max} \tag{25}$$

transforms the entire sphere into region Z.

Now, the zone vector spherical functions (ZVSF) are introduced as

$$\hat{\mathbf{V}}_j(l,b) = \sqrt{\alpha} K_j(l,\hat{b}) \boldsymbol{e}_r. \tag{26}$$

$$\hat{\mathbf{T}}_j(l,\hat{b}) = \sqrt{\alpha}(T_j^l(l,\hat{b})\boldsymbol{e}_l + T_j^b(l,\hat{b})\boldsymbol{e}_b), \tag{27}$$

$$\hat{\mathbf{S}}_j(l,\hat{b}) = \sqrt{\alpha}(S_j^l(l,\hat{b})\boldsymbol{e}_l + S_j^b(l,\hat{b})\boldsymbol{e}_b). \tag{28}$$

These functions are orthonormal on the set Z, so the following relations are valid:

$$\int_Z \left(\hat{\mathbf{V}}_i \cdot \hat{\mathbf{V}}_j\right)\mathrm{d}\omega = \int_Z \left(\hat{\mathbf{T}}_i \cdot \hat{\mathbf{T}}_j\right)\mathrm{d}\omega = $$
$$= \int_Z \left(\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j\right)\mathrm{d}\omega = \begin{cases} 0, & i \neq j, \\ 1, & i = j; \end{cases} \tag{29}$$

$$\int_Z \left(\hat{\mathbf{V}}_i \cdot \hat{\mathbf{T}}_j\right)\mathrm{d}\omega = \int_Z \left(\hat{\mathbf{V}}_i \cdot \hat{\mathbf{S}}_j\right)\mathrm{d}\omega = $$
$$= \int_Z \left(\hat{\mathbf{S}}_i \cdot \hat{\mathbf{T}}_j\right)\mathrm{d}\omega = 0,\ \forall i,j. \tag{30}$$

where, for example,

$$\int_Z \left(\hat{\mathbf{T}}_i \cdot \hat{\mathbf{T}}_j\right)\mathrm{d}\omega = $$

$$\alpha \int_0^{2\pi} \mathrm{d}l \int_{b_{\min}}^{b_{\max}} T_i^l(l,\hat{b})\, T_j^l(l,\hat{b}) \cos b\,\mathrm{d}b\ + $$
$$\alpha \int_0^{2\pi} \mathrm{d}l \int_{b_{\min}}^{b_{\max}} T^b{}_i(l,\hat{b})\, T_j^b(l,\hat{b}) \cos b\,\mathrm{d}b\,. \tag{31}$$

Since the radial velocities are not available in the catalogue UCAC4, in what follows, we consider only the tangential stellar velocity field specified in region Z on the celestial sphere:

$$\mathbf{U}(l,b) = \mu_l^* \, \boldsymbol{e}_l + \mu_b^* \, \boldsymbol{e}_b, \tag{32}$$

where $\mu_l^* = \mathcal{K}\mu_l \cos b$; $\mu_b^* = \mathcal{K}\mu_b$.

We can now use the system of Zone Vector Spherical Functions to decompose the velocity field as

$$\mathbf{U}(l,b) = \sum_j t_j \hat{\mathbf{T}}_j(l,\hat{b}) + \sum_j s_j \hat{\mathbf{S}}_j(l,\hat{b}). \tag{33}$$

Given the orthonormality of the basis, the decomposition coefficients can be calculated by the following formulas:

$$t_j = \int_Z \left(\mathbf{U}\cdot\hat{\mathbf{T}}_j\right) \mathrm{d}\omega; \quad s_j = \int_Z \left(\mathbf{U}\cdot\hat{\mathbf{S}}_j\right) \mathrm{d}\omega. \tag{34}$$

Note that the expressions (33) and (34) are valid for all sky analysis since in this case $\alpha = 1$ and $\hat{\mathbf{T}}_j(l,\hat{b})$ and $\hat{\mathbf{S}}_j(l,\hat{b})$ become $\mathbf{T}_j(l,b)$ and $\mathbf{S}_j(l,b)$ respectively.

6 The method in practice

Assume that we have at our disposal a catalogue of stars with Galactic coordinates and proper motion components in latitude and longitude. As was mentioned before, the full and the zonal catalogue may be treated likewise, so let us describe the sequence of steps for the kinematic analysis of the stellar velocity field using ZVSH.

(1) Calculating the ZVSH decomposition coefficients t_j, s_j of the velocity field. These coefficients and their root-mean-square (rms) errors can be derived from the equations

$$\mu_l^* = \sum_j t_j \hat{\mathbf{T}^l}_j(l,\hat{b}) + \sum_j s_j \hat{\mathbf{S}^l}_j(l,\hat{b}), \tag{35}$$

$$\mu_b^* = \sum_j t_j \hat{\mathbf{T}^b}_j(l,\hat{b}) + \sum_j s_j \hat{\mathbf{S}^b}_j(l,\hat{b}) \tag{36}$$

by the standard least-squares procedure. The total number of decomposition terms can be chosen from the condition that the residuals in the velocity field components with statistically significant harmonics subtracted from them behave as random quantities (Brosche 1966; Mignard & Klioner 2012).

It is necessary to emphasize that Eqs. (4) give us a physical model of the stellar velocity field since we know the physical meaning of each parameter of it. However, this model is not complete, because it does not incorporate all physical content of the observed data. In contrast to that, Eqs. (35) and (36) are complete, because all the information of the data is captured by the decomposition coefficients (due to completeness of the VSF and ZVSF). However, this model is not physical, because we do not know the underlying physics of each decomposition coefficient.

(2) Determining the parameters of a specific kinematic model. Once the decomposition coefficients $t_j \pm \sigma_{t_j}, s_j \pm \sigma_{s_j}$ have been determined, we can write the equations relating the decomposition coefficients to the sought for model parameters. When the full catalogue is used, the physical meaning of the coefficients up to $n \leq 2$ is shown in Tables 5 and 6. From these formulas the kinematic parameters of the Ogorodnikov-Milne model may be obtained easily.

In case of zonal catalogue the relation between the decomposition coefficients and kinematic parameters depends on the size of the zone. In this paper we analyze the kinematics of the northern and southern Galactic hemispheres, and the relation between the decomposition coefficients and kinematic parameters is more complicated, as is evident from Table 1. To determine the kinematic parameters, the number of such equations is taken to be equal to the number of parameters. Thus several (theoretically infinitely many) estimates of model parameters can be obtained. In practice, it is appropriate to construct the solutions for the lowest-order decomposition terms. In our method, we will use two estimates of the parameters, which we refer to as the main and alternative solutions:

$$\begin{bmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ M_{13}^+ \\ M_{23}^+ \\ M_{12}^+ \\ M_{11}^* \\ M_{33}^* \end{bmatrix} = \mathbf{A} \begin{bmatrix} s_{101} \\ s_{110} \\ s_{111} \\ s_{201} \\ s_{210} \\ s_{211} \\ s_{220} \\ s_{221} \\ t_{101} \\ t_{110} \\ t_{111} \end{bmatrix}, \quad \begin{bmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ M_{13}^+ \\ M_{23}^+ \\ M_{12}^+ \\ M_{11}^* \\ M_{33}^* \end{bmatrix} = \mathbf{B} \begin{bmatrix} s_{101} \\ s_{110} \\ s_{111} \\ s_{301} \\ t_{201} \\ t_{110} \\ t_{111} \\ t_{210} \\ t_{211} \\ t_{220} \\ t_{221} \end{bmatrix}. \tag{37}$$

Here $\bar{U}, \bar{V}, \bar{W}$ are the mean values of the products $U\pi, V\pi, W\pi$ if the parallaxes π are unknown, and $M_{11}^* = M_{11}^+ - M_{22}^+$, $M_{33}^* = M_{33}^+ - M_{22}^+$ since the value M_{22}^+ can not be determined when only the PM are available (Clube 1972). The matrices $\mathbf{A}$ and $\mathbf{B}$ are shown in Tables 2 and 3 (Vityazev & Tsvetkov 2011).

(3) Analyzing the decomposition coefficients not described by the model. In the case of all-sky analysis, the Ogorodnikov-Milne model can be completely described by the decomposition coefficients t_{npk}, s_{npk} up to $k \leq 2$. All of the remaining decomposition terms with significant coefficients define the systematic components of the stellar velocity field that are not incorporated in the standard model. Establishing the physical meaning of these harmonics is a separate problem that basically reduces to constructing a new kinematic model.

7 The all-sky proper-motion analysis, extra-model harmonics

We applied the VSF to study the kinematics of the UCAC4 proper motions in different 1^{m} bins using the UCAC fit model magnitudes (579–642 nm). The following samples have been selected:

Table 1 The kinematics of the ZVSF (up to $n \leq 3$) in the frame of Ogorodnikov-Milne model. The upper and lower signs correspond to the northern and southern hemisphere, respectively. In cases of a single sign the signs of both hemispheres coincide. Units are km s^{-1} kpc^{-1}.

j	n	k	l	T_j	S_j
1	1	0	1	$+1.949\,\Omega_3$	$-1.949\bar{W} \pm 0.873(M^*_{33} - \frac{1}{2}M^*_{11})$
2	1	1	0	$\mp0.768\bar{U} + 1.791\,\Omega_2 - 0.256M^+_{13}$	$-1.791\bar{V} \mp 0.768\,\Omega_1 \pm 1.279M^+_{23}$
3	1	1	1	$\pm0.768\bar{V} + 1.791\,\Omega_1 + 0.256M^+_{23}$	$-1.791\bar{U} \mp 0.768\,\Omega_2 \pm 1.279M^+_{13}$
4	2	0	1	$\mp0.453\,\Omega_3$	$\pm0.453W + 0.274(M^*_{33} - \frac{1}{2}M^*_{11})$
5	2	1	0	$+0.332\bar{U} \mp 0.332\,\Omega_2 \pm 0.332M^+_{13}$	$\pm0.332\bar{V} + 0.332\,\Omega_1 + 0.728M^+_{23}$
6	2	1	1	$-0.332\bar{V} \mp 0.332\,\Omega_1 \mp 0.332M^+_{23}$	$\pm0.332\bar{U} - 0.332\,\Omega_2 + 0.728M^+_{13}$
7	2	2	0	$\pm0.216M^*_{11}$	$+1.338M^+_{12}$
8	2	2	1	$\mp0.433M^+_{12}$	$+0.669M^*_{11}$
9	3	0	1	$+0.270\,\Omega_3$	$-0.270\bar{W} \mp 0.017(M^*_{33} - \frac{1}{2}M^*_{11})$
10	3	1	0	$\mp0.199\bar{U} + 0.199\,\Omega_2 - 0.199M^+_{13}$	$-0.199\bar{V} \mp 0.199\,\Omega_1 \mp 0.199M^+_{23}$
11	3	1	1	$\pm0.199\bar{V} + 0.199\,\Omega_1 + 0.199M^+_{23}$	$-0.199\bar{U} \pm 0.199\,\Omega_2 \mp 0.199M^+_{13}$
12	3	2	0	$-0.109M^*_{11}$	$\mp0.463M^+_{12}$
13	3	2	1	$+0.219M^+_{12}$	$\mp0.231M^*_{11}$
14	3	3	0	0	0
15	3	3	1	0	0

Table 2 Matrix **A** for calculating the main solution (37). The upper and lower signs correspond to the northern and southern hemispheres, respectively; if there is one sign, then the signs of the coefficient are identical for the northern and southern hemispheres.

0	0	-0.54	0	0	±1.10	0	0	0	±0.44	0
0	-0.54	0	0	±1.10	0	0	0	0	0	∓0.44
-0.29	0	0	±0.94	0	0	0	0	0	0	0
0	±0.21	0	0	-0.64	0	0	0	0	0	0.77
0	0	∓0.21	0	0	0.64	0	0	0	0.77	0
0	0	0	0	0	0	0	0	0.51	0	0
0	0	±0.15	0	0	1.16	0	0	0	0.15	0
0	±0.15	0	0	1.16	0	0	0	0	0	-0.15
0	0	0	0	0	0	0.75	0	0	0	0
0	0	0	0	0	0	0	1.49	0	0	0
±0.49	0	0	2.10	0	0	0	0.75	0	0	0

Table 3 Matrix **B** for calculating the main solution (37). The upper and lower signs correspond to the northern and southern hemispheres, respectively; if there is one sign, then the signs of the coefficient are identical for the northern and southern hemispheres.

0	0	-0.42	0	0	±0.56	0	2.04	0	0	0
0	-0.42	0	0	0	0	∓0.56	0	-2.04	0	0
-0.06	0	0	-3.25	0	0	0	0	0	0	0
0	±0.14	0	0	0	0	0.84	0	±1.18	0	0
0	0	∓0.14	0	0	0.84	0	±1.18	0	0	0
0	0	0	0	∓2.21	0	0	0	0	0	0
0	0	±0.28	0	0	0.28	0	±2.15	0	0	0
0	±0.28	0	0	0	0	-0.28	0	∓2.15	0	0
0	0	0	0	0	0	0	0	0	0	∓2.31
0	0	0	0	0	0	0	0	0	±4.63	0
±1.00	0	0	∓7.25	0	0	0	0	0	±2.32	0

$11^{\rm m}$ (1 586 674 stars), $12^{\rm m}$ (3 757 366 stars), $13^{\rm m}$ (8 196 861 stars), $14^{\rm m}$ (17 805 026 stars), $15^{\rm m}$ (36 204 620 stars), $16^{\rm m}$ (33 440 398 stars).

The volume of processed data for each sample is very large, and hence determination of the VSH decomposition coefficients is a real challenge. This difficulty can be overcome by data pre-pixelization on the sphere. As applied to our problem, the pixelization scheme should satisfy the requirement that the pixel centers are equidistant in both latitude and longitude. Two schemes satisfy this requirement. One of them is HEALPix (Górski et al. 2005). The other is the so-called Equidistant Cylindrical Projection (ECP). Pixelization algorithms were discussed in detail previously (Vityazev & Tsvetkov 2009). In this paper, we focussed on ECP, in which the stellar proper motions are averaged over spherical trapezia obtained by a uniform partitioning of the equator and the latitude circle into $M = 180$ and $N = 90$ parts, respectively.

Table 4 The Ogorodnikov-Milne parameters derived from the UCAC4 catalogue by LSM-solution over the entire sphere. Here A and B are the Oort constants (km s^{-1} kpc^{-1}); P, the period of the Galactic rotation in the vicinity of the Sun in million years; L and B, the longitude and latitude of the solar motion apex (degree), respectively.

	$11^{\rm m}$	$12^{\rm m}$	$13^{\rm m}$	$14^{\rm m}$	$15^{\rm m}$	$16^{\rm m}$
$\bar{U}$	18.3 ± 0.2	16.9 ± 0.1	14.9 ± 0.1	12.6 ± 0.1	10.6 ± 0.1	9.6 ± 0.1
$\bar{V}$	45.0 ± 0.2	40.6 ± 0.1	36.3 ± 0.1	34.3 ± 0.1	32.3 ± 0.1	30.8 ± 0.1
$\bar{W}$	13.8 ± 0.2	11.1 ± 0.1	8.8 ± 0.1	7.8 ± 0.1	6.2 ± 0.1	5.0 ± 0.1
Ω_1	0.7 ± 0.2	0.6 ± 0.1	0.1 ± 0.1	-0.8 ± 0.1	-0.8 ± 0.1	-0.4 ± 0.1
Ω_2	-4.6 ± 0.2	-4.6 ± 0.1	-3.0 ± 0.1	-1.2 ± 0.1	-0.6 ± 0.1	-0.6 ± 0.1
$B = \Omega_3$	-10.0 ± 0.2	-10.7 ± 0.1	-11.4 ± 0.1	-11.0 ± 0.1	-10.9 ± 0.1	-10.9 ± 0.1
M_{13}^+	-0.4 ± 0.2	-0.6 ± 0.2	-0.5 ± 0.2	0.3 ± 0.2	0.1 ± 0.1	-0.2 ± 0.1
M_{23}^+	-0.8 ± 0.2	0.6 ± 0.2	1.6 ± 0.2	1.7 ± 0.2	1.6 ± 0.1	1.9 ± 0.1
$A = M_{12}^+$	14.2 ± 0.2	13.3 ± 0.2	11.4 ± 0.2	10.8 ± 0.2	10.2 ± 0.1	9.1 ± 0.1
M_{11}^*	-1.7 ± 0.4	-2.5 ± 0.4	-2.8 ± 0.3	-2.8 ± 0.3	-4.4 ± 0.3	-4.4 ± 0.3
M_{33}^*	0.7 ± 0.4	0.2 ± 0.4	-0.3 ± 0.3	-0.3 ± 0.3	-1.1 ± 0.3	-1.4 ± 0.3
$A + B$	4.2 ± 0.3	2.6 ± 0.2	0.0 ± 0.2	-0.3 ± 0.2	-0.7 ± 0.2	-1.8 ± 0.1
$A - B$	24.2 ± 0.3	24.0 ± 0.2	22.8 ± 0.2	21.9 ± 0.2	21.1 ± 0.2	21.1 ± 0.1
P	253.9 ± 3.0	256.0 ± 2.4	269.5 ± 2.7	280.5 ± 2.9	291.2 ± 2.0	307.2 ± 2.2
L	67.9 ± 0.2	$67,4 \pm 0.2$	67.6 ± 0.2	69.9 ± 0.2	71.8 ± 0.2	72.7 ± 0.2
B	15.9 ± 0.2	14.3 ± 0.2	12.6 ± 0.2	12.1 ± 0.2	10.5 ± 0.2	$8,8 \pm 0.2$

First of all, we performed the traditional combined LSM solutions for $\mu_l \cos b$ and μ_b from Eq. (4). The results are listed in Table 4. As is evident from this table, the main features of the stellar kinematics that can be derived from the data are the Oort constants A and B; the angular speed of the local Galactic rotation $A - B$; the slope of the local rotational velocity curve $A + B$; the period of the Galaxy rotation in the vicinity of the Sun as well as the longitude L and latitude B of the solar motion apex.

We then performed the VSF analysis of the same samples of stars. The results are listed in Tables 5 and 6. All the harmonics associated with the Ogorodnikov-Milne model yield practically the same values of the kinematic parameters as those listed in Table 4. Upon close examination, we see that the VSF method gives not only the information which we usually get from the traditional LSM technique, but yields some extra information. Indeed, apart from the expected model harmonics it detected several extra-model harmonics among which the t_{211} and s_{310} coefficients are very powerful.

8 The proper motions analysis in the northern and southern hemispheres

We applied the ZVSHs formalism to investigate the proper motions of stars in the northern and southern Galactic hemispheres based on the data from the UCAC4 catalogue. The stars were sampled within $1^{\rm m}$ bins with averages $11^{\rm m}$ (743 246 northern and 844 299 southern stars), $12^{\rm m}$ (1 729 824 northern and 2 028 132 southern stars), $13^{\rm m}$ (3 719 008 northern and 4 478 186 southern stars), $14^{\rm m}$ (7 929 146 northern and 9 876 006 southern stars), $15^{\rm m}$ (15 891 947 northern and 20 312 698 southern stars), $16^{\rm m}$ (15 030 372 northern and 18 410 068 southern stars).

The most striking result that we obtained when analyzing the velocity field in different hemispheres is that the statistically reliable values of Ω_1 and M_{32}^+ have different signs in different hemispheres. As can be seen from Tables 7 and 8, the reality of these parameters is confirmed by the excellent agreement between the main and alternative solutions, where almost complete coincidence is observed within the error limits of these estimates. We see that Ω_1 and M_{32}^+ are almost the same in magnitude but different in sign and therefore when the kinematic parameters are traditionally determined by the LSM from the data over the entire celestial sphere the resulting Ω_1 and M_{32}^+ are quite small (Table 4).

9 The vertical gradient of the Galaxy's rotational velocity

Application of VSF and ZVSF to UCAC4 proper motions yielded two unexpected results: large extra-model harmonics and different signs of the parameters Ω_1 and M_{23}^+ in the northern and southern Galactic hemispheres. The existence of harmonics S_{310} and T_{211} was first reported by Vityazev & Shuksto (2005), and confirmed by Makarov & Murphy (2007) and Vityazev & Tsvetkov (2009). The second effect was detected by Vityazev & Tsvetkov (2012). Both effects need explanation. For this purpose consider the contribution of the Ω_1 and M_{23}^+ terms to the proper motions

$$\mu_l^*(l, b) = -\Omega_1 \sin b \cos l + M_{23}^+ \sin b \cos l, \tag{38}$$

$$\mu_b^*(l, b) = \Omega_1 \sin l + M_{23}^+ \cos 2b \cos l, \tag{39}$$

where the numerical values of Ω_1 and M_{23}^+ coincide with the mean values $\Omega_1 = 19.55$ and $M_{23}^+ = -18.07$ km s^{-1} kpc^{-1} for the northern hemisphere and $\Omega_1 = -22.61$ and $M_{23}^+ = 21.48$ km s^{-1} kpc^{-1} for the southern hemisphere.

Table 5 Decomposition of the UCAC4 proper motions into VSF over the entire sphere. The toroidal coefficients t_j and their relation to the parameters of the Ogorodnikov-Milne model. Units are km s^{-1} kpc^{-1}.

Model Harmonics						
t_j	11^{m}	12^{m}	13^{m}	14^{m}	15^{m}	16^{m}
$t_{101} = 2.89\,\omega_3$	-29.3	-31.1	-33.0	-32.0	-31.6	-31.5
$t_{110} = 2.89\,\omega_2$	-13.4	-13.3	-8.6	-3.4	-1.7	-1.7
$t_{111} = 2.89\,\omega_1$	1.8	1.7	0.2	-2.4	-2.3	-1.2
Extra-Model Harmonics						
t_{201}	1.7	1.8	1.3	-0.3	0.2	0.0
t_{210}	-11.1	-11.9	-10.4	-7.4	-5.1	-3.5
t_{211}	31.2	30.6	29.6	28.0	27.4	27.4
σ	±0.4	±0.3	±0.3	±0.3	±0.2	±0.2

Table 6 Decomposition of the UCAC4 proper motions in VSF over the entire sphere. The values of spheroidal coefficients s_j and their relation to the parameters of the Ogorodnikov-Milne model. Units are in km s^{-1} kpc^{-1}.

Model Harmonics						
s_j	11^{m}	12^{m}	13^{m}	14^{m}	15^{m}	16^{m}
$s_{101} = -2.89\,\bar{W}$	-40.0	-32.4	-25.4	-22.5	-18.1	$-14,5$
$s_{110} = -2.89\,\bar{V}$	-130.0	-117.0	-105.0	-99.2	-93.5	-89.2
$s_{111} = -2.89\,\bar{U}$	-53.1	-48.8	-43.2	-36.4	-30.8	-27.8
$s_{201} = 1.30(M^*_{33} - \frac{1}{2}M^*_{11})$	2.3	2.0	1.6	1.4	1.5	1.0
$s_{210} = 2.24\,M^+_{23}$	-1.9	1.3	3.7	3.8	3.5	4.3
$s_{211} = 2.24\,M^+_{13}$	-1.0	-1.3	-1.1	0.7	0.3	-0.5
$s_{220} = 2.24\,M^+_{12}$	31.6	29.6	25.5	24.2	22.9	20.4
$s_{221} = 1.12\,M^*_{11}$	-1.6	-2.7	-3.1	-3.1	-5.0	-4.9
Extra-Model Harmonics						
s_{301}	-9.5	-7.9	-6.0	-6.1	-6.2	-6.1
s_{310}	-15.7	-15.5	-16.5	-16.3	-16.0	-16.5
s_{311}	-4.3	-5.6	-4.9	-4.5	-3.8	-3.4
σ	±0.4	±0.3	±0.3	±0.3	±0.2	±0.2

From Eqs. (33) and (34) applied to the entire sphere we find $s_{310} = -15.3$; $t_{211} = 29.5$. Comparing these estimates with the mean values $s_{310} = -16.1$; $t_{211} = 29.0$ from Tables 5–6 gives grounds to say that both above mentioned unexpected results are connected in such a way that the values of Ω_1 and M^+_{23} with different signs in different hemispheres yield extra-model harmonics in proper motions when analyzed by VSF on the entire sphere.

Now, in the galactocentric cylindrical coordinate system we have (Miyamoto et al. 1993)

$$\Omega_1 - M^+_{32} = -\frac{\partial V_{\mathrm{S}}}{\partial z}, \tag{40}$$

where V_{S} is the circular velocity of the motion of the local reference frame around the Galactic center. This quantity is identified with the Galaxy's rotational velocity in the solar neighborhood. Table 9 gives the numerical values for the left-hand side of Eq. (40) that we obtained from different samples of the UCAC4 catalog. We see from this table that the vertical gradient of the Galaxy's rotational velocity $\frac{\partial V_{\mathrm{S}}}{\partial z}$ has different signs in the northern and southern Galactic hemispheres, with the velocity decreasing with increasing distance from the principal Galactic plane. It is also important that the magnitudes of the gradient are approximately identical for all the UCAC4 samples. Averaging these results over both hemispheres leads us to conclude that the magnitude of the vertical gradient of the Galaxys rotational lag for the stars from 11^{m} to 16^{m} is determined very reliably and lies within the range

$$(39.0 \pm 0.2) < \left|\frac{\partial V_{\mathrm{S}}}{\partial z}\right| < (41.2 \pm 0.3) \quad \mathrm{km\,s^{-1}\,kpc}. \tag{41}$$

Note that the lower bound was calculated as the average of three faint-star bins (14^{m}–16^{m}) while the upper bound corresponds to bright (11^{m}–13^{m}) UCAC4 stars.

There is extensive literature devoted to determining the vertical gradient of the Galaxy's rotational velocity (Majewski 1993; Girard 2006). Note, however, that the first studies of the Galaxy's rotational lag were carried out by various indirect methods. For example, Hanson (1989) proceeded from the increase of the component V of solar motion relative to stars with their distances from the Galactic plane, and found the gradient to be 30 km s^{-1} kpc^{-1} for the Galactic thick disk (1–4 kpc). Based on the overall Galactic potential model, Girard (2006) offered a dynamical explanation for the Galaxy's rotational lag. Makarov & Murphy (2007) hypothesized that the rotational lag also exists in the thin

Table 7 Main and alternative solutions for the northern Galactic hemisphere stars of the UCAC4. Units are km s^{-1} kpc^{-1}.

	11^{m}	12^{m}	13^{m}	14^{m}	15^{m}	16^{m}
Main Solution						
$\bar{U}$	12.80 ± 0.7	8.9 ± 0.6	8.1 ± 0.4	7.3 ± 0.4	6.9 ± 0.3	6.8 ± 0.3
$\bar{V}$	22.06 ± 0.7	18.5 ± 0.6	15.0 ± 0.4	12.9 ± 0.4	11.4 ± 0.3	9.4 ± 0.3
$\bar{W}$	7.76 ± 0.5	6.4 ± 0.4	5.7 ± 0.3	4.9 ± 0.3	3.6 ± 0.2	2.6 ± 0.2
Ω_1	21.71 ± 0.5	20.8 ± 0.5	19.6 ± 0.3	18.5 ± 0.3	18.1 ± 0.2	18.7 ± 0.2
Ω_2	-10.47 ± 0.5	-12.0 ± 0.5	-9.4 ± 0.3	-5.8 ± 0.3	-3.7 ± 0.2	-2.8 ± 0.2
Ω_3	-9.70 ± 0.3	-10.5 ± 0.2	-11.4 ± 0.2	-11.7 ± 0.1	-11.4 ± 0.1	-11.2 ± 0.1
M_{13}^{+}	-5.17 ± 0.6	-7.8 ± 0.5	-6.8 ± 0.4	-5.1 ± 0.3	-4.1 ± 0.2	-3.6 ± 0.2
M_{23}^{+}	-20.72 ± 0.6	-18.8 ± 0.5	-17.4 ± 0.4	-17.2 ± 0.3	-17.2 ± 0.2	-17.3 ± 0.2
M_{12}^{+}	11.92 ± 0.4	12.0 ± 0.3	11.1 ± 0.2	10.3 ± 0.2	10.1 ± 0.1	9.0 ± 0.2
M_{11}^{*}	-1.76 ± 0.7	-1.2 ± 0.7	-2.2 ± 0.5	-3.3 ± 0.4	-6.1 ± 0.3	-6.4 ± 0.3
M_{33}^{*}	-14.48 ± 1.1	-11.0 ± 1.0	-8.3 ± 0.7	-8.7 ± 0.6	-9.8 ± 0.4	-9.7 ± 0.5
Alternative Solution						
$\bar{U}$	11.58 ± 1.1	10.9 ± 1.0	12.2 ± 0.7	14.2 ± 0.6	13.3 ± 0.4	11.8 ± 0.4
$\bar{V}$	18.72 ± 1.1	16.6 ± 1.0	14.1 ± 0.7	11.6 ± 0.6	11.2 ± 0.4	8.6 ± 0.4
$\bar{W}$	7.98 ± 1.6	4.7 ± 1.5	2.5 ± 1.0	0.4 ± 0.9	-1.3 ± 0.6	-2.1 ± 0.7
Ω_1	23.76 ± 0.7	22.0 ± 0.7	20.2 ± 0.4	19.3 ± 0.4	18.3 ± 0.3	19.2 ± 0.3
Ω_2	-11.23 ± 0.7	-10.8 ± 0.7	-7.1 ± 0.4	-1.9 ± 0.4	0.0 ± 0.3	0.1 ± 0.3
Ω_3	-11.28 ± 1.1	-10.4 ± 1.0	-9.0 ± 0.7	-9.1 ± 0.6	-9.5 ± 0.4	-11.3 ± 0.5
M_{13}^{+}	-6.49 ± 1.1	-5.7 ± 1.0	-2.5 ± 0.7	2.2 ± 0.6	2.6 ± 0.4	1.6 ± 0.5
M_{23}^{+}	-24.36 ± 1.1	-20.8 ± 1.0	-18.4 ± 0.7	-18.7 ± 0.6	-17.4 ± 0.4	-18.2 ± 0.5
M_{12}^{+}	11.26 ± 1.2	8.7 ± 1.0	8.9 ± 0.7	10.4 ± 0.6	10.7 ± 0.5	10.1 ± 0.5
M_{11}^{*}	-1.31 ± 2.3	2.6 ± 2.1	-0.2 ± 1.4	-7.2 ± 1.3	-12.8 ± 0.9	-14.2 ± 1.0
M_{33}^{*}	-13.64 ± 3.8	-12.7 ± 3.4	-14.3 ± 2.3	-20.7 ± 2.1	-23.9 ± 1.5	-23.9 ± 1.6

Table 8 Main and alternative solutions for the **southern** Galactic hemisphere stars of the UCAC4. Units are km s^{-1} kpc^{-1}.

	11^{m}	12^{m}	13^{m}	14^{m}	15^{m}	16^{m}
Main Solution						
$\bar{U}$	9.0 ± 0.7	5.5 ± 0.9	6.0 ± 0.7	4.6 ± 0.6	4.8 ± 0.4	4.8 ± 0.3
$\bar{V}$	20.5 ± 0.7	16.3 ± 0.9	11.8 ± 0.7	11.3 ± 0.6	9.7 ± 0.4	8.5 ± 0.3
$\bar{W}$	8.0 ± 0.5	6.0 ± 0.7	4.9 ± 0.5	3.6 ± 0.4	2.0 ± 0.3	0.6 ± 0.2
Ω_1	-21.4 ± 0.6	-21.5 ± 0.7	-21.7 ± 0.5	-21.2 ± 0.4	-20.7 ± 0.3	-20.1 ± 0.2
Ω_2	4.6 ± 0.6	5.0 ± 0.7	5.1 ± 0.5	5.6 ± 0.4	4.4 ± 0.3	3.3 ± 0.2
Ω_3	-11.0 ± 0.3	-12.2 ± 0.4	-12.2 ± 0.3	-11.3 ± 0.2	-11.3 ± 0.1	-11.0 ± 0.1
M_{13}^{+}	6.7 ± 0.6	8.6 ± 0.8	6.7 ± 0.6	6.9 ± 0.5	5.1 ± 0.3	4.0 ± 0.3
M_{23}^{+}	20.1 ± 0.6	20.5 ± 0.8	22.9 ± 0.6	22.0 ± 0.5	21.5 ± 0.3	21.8 ± 0.3
M_{12}^{+}	15.8 ± 0.4	14.2 ± 0.5	12.3 ± 0.4	11.4 ± 0.3	10.9 ± 0.2	9.5 ± 0.2
M_{11}^{*}	-2.4 ± 0.8	-5.6 ± 1.1	-4.4 ± 0.8	-2.7 ± 0.6	-2.4 ± 0.4	-1.7 ± 0.3
M_{33}^{*}	14.4 ± 1.2	13.0 ± 1.6	8.9 ± 1.2	10.6 ± 1.0	10.7 ± 0.6	11.0 ± 0.5
Alternative Solution						
$\bar{U}$	10.3 ± 1.2	6.6 ± 1.5	4.9 ± 1.1	5.0 ± 0.9	4.0 ± 0.6	4.8 ± 0.5
$\bar{V}$	19.4 ± 1.2	10.7 ± 1.5	10.3 ± 1.1	11.2 ± 0.9	8.5 ± 0.6	6.8 ± 0.5
$\bar{W}$	10.9 ± 1.8	3.9 ± 2.3	5.8 ± 1.7	3.6 ± 1.4	3.9 ± 0.9	3.5 ± 0.7
Ω_1	-22.1 ± 0.8	-24.9 ± 1.0	-22.6 ± 0.8	-21.4 ± 0.6	-21.5 ± 0.4	-21.1 ± 0.3
Ω_2	3.8 ± 0.8	4.4 ± 1.0	5.8 ± 0.8	5.4 ± 0.6	4.9 ± 0.4	3.3 ± 0.3
Ω_3	-10.5 ± 1.2	-8.0 ± 1.6	-7.2 ± 1.2	-7.5 ± 0.9	-6.3 ± 0.6	-8.6 ± 0.5
M_{13}^{+}	5.4 ± 1.2	7.6 ± 1.5	8.0 ± 1.2	6.5 ± 0.9	5.9 ± 0.6	4.1 ± 0.5
M_{23}^{+}	21.3 ± 1.2	26.5 ± 1.5	24.6 ± 1.2	22.2 ± 0.9	22.9 ± 0.6	23.7 ± 0.5
M_{12}^{+}	16.3 ± 1.3	17.9 ± 1.6	15.5 ± 1.2	13.4 ± 1.0	11.9 ± 0.7	10.9 ± 0.5
M_{11}^{*}	2.5 ± 2.5	-6.8 ± 3.3	-9.2 ± 2.5	-5.2 ± 2.0	-4.1 ± 1.3	0.1 ± 1.0
M_{33}^{*}	10.2 ± 4.2	17.0 ± 5.4	4.5 ± 4.1	9.5 ± 3.3	5.6 ± 2.2	5.3 ± 1.7

disk (200–300 pc). They found the gradient from Hipparcos data to be 20 km s^{-1} kpc^{-1}, which is completely confirmed by our results (Vityazev & Tsvetkov 2012). In contrast to the indirect methods listed above, our ZVSF approach can

Table 9 Numerical values for the left-hand side of (40) obtained from the northern and southern Galactic hemispheres of the UCAC4. Units are $\mathrm{km\,s^{-1}\,kpc^{-1}}$.

Catalogue UCAC4, Main Solution						
	11^{m}	12^{m}	13^{m}	14^{m}	15^{m}	16^{m}
$(\Omega_1 - M_{23}^{+})_{\mathrm{N}}$	42.4 ± 0.8	39.6 ± 0.7	36.9 ± 0.5	35.7 ± 0.4	35.2 ± 0.3	36.0 ± 0.3
$(\Omega_1 - M_{23}^{+})_{\mathrm{S}}$	-41.5 ± 0.9	-42.1 ± 1.1	-44.6 ± 0.8	-43.2 ± 0.7	-42.3 ± 0.4	-41.9 ± 0.3
$\left\lvert \frac{\partial V_{\mathrm{S}}}{\partial z} \right\rvert$	42.0 ± 0.6	40.8 ± 0.7	40.8 ± 0.5	39.4 ± 0.4	38.7 ± 0.3	39.0 ± 0.2

be classified as a direct method, because it detects the vertical velocity gradient by analyzing the parameters of the Ogorodnikov-Milne model applied separately to the northern and southern Galactic hemispheres, where this gradient retains its sign.

10 Conclusion

The main result of this paper is the development of a method for kinematic studies of the proper motions of stars based on the use of vector spherical functions. Two approaches are proposed - full sphere and half sphere analysis. Applications of both techniques to the data from recent UCAC4 catalogue gave two kinds of information. The first one is quite expected since it could be obtained with the traditional LSM method applied for estimating kinematic parameters of the 3-D Ogorodnikov-Milne model. As for these results, we may say that the coordinates of the solar motion apex, the Oort constants A and B; the angular speed of the local Galactic rotation $A - B$; the slope of the local rotational velocity curve $A + B$ have been derived from stars from 11^{m} to 16^{m}. These estimates are of specific interest since they show very noticeable trends when we go from bright to faint stars (Table 4).

The second kind of results is unusual for traditional techniques. Indeed, application of VSF to the data distributed over the entire sphere resulted in the detection of significant harmonics that can not be explained by the standard model. Furthermore, the zonal vector spherical function used for the data in the northern and southern hemispheres gave the second unexpected result: that of the reversal of the signs of the parameters Ω_1 and M_{23}^{+} when passing from one hemispheres to another. While attempting to explain both effects we succeeded to show that they show up simultaneously and thus have one and the same physical origin. An analysis of the difference of Ω_1 and M_{23}^{+} in the galactocentric coordinate system indicates that both detected effects are associated with retardation of the speed of the Galaxy rotation with increasing distance from the principal Galactic plane. We derived the numerical estimation $\sim$40 $\mathrm{km\,s^{-1}\,kpc^{-1}}$, which agrees well with our previous estimates based on Hipparcos, Tycho-2, and UCAC3 catalogues (Vityazev & Tsvetkov 2009, 2011, 2012). However, we do not rule out other possible explanations for the extra-model harmonics on the full sphere in the kinematic analysis with vector spherical functions.

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