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Properties of the Tycho-2 Catalogue from Gaia Data Release

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Abstract—Based on the measurements performed in the first 14 months of Gaia operation, we have solved the problem of obtaining the systematic differences between the stellar positions and proper motions of the TGAS (Tycho—Gaia Astrometric Solution) and Tycho-2 catalogues. By dividing the common stars from the TGAS and Tycho-2 catalogues into three G-magnitude groups for mean values of $10\overset{m}{.}5$, $11\overset{m}{.}5$, and $13\overset{m}{.}0$, we have obtained the systematic differences between the stellar equatorial coordinates and proper motions of both catalogues in the form of a decomposition into vector spherical harmonics by taking into account the magnitude equation. The systematic components have been extracted from the individual differences with a probability of 0.977–0.999. The constructed model of systematic differences allows any position measurements performed using Tycho-2 as a reference catalogue to be transformed to the TGAS frame. An important fact is the existence of a magnitude equation in the systematic differences: when passing from bright ($G = 10^m$) to faint ($G = 13^m$) stars, the systematic position differences change within the range from approximately −40 to 15 mas, while the systematic proper motion differences change from −3 to 3 mas yr^{-1}. The orientation and mutual rotation parameters of the Tycho-2 and TGAS frames have also been found to be different for stars of different magnitudes: when passing from bright to faint stars, the rotation angle of the Tycho-2 frame relative to TGAS changes from 3.51 to 5.63 mas, while the angular velocity of rotation changes from 0.35 to 1.22 mas yr^{-1}. Based on the developed method that allows the extent to which the systematic errors in the equatorial proper motions of stars affect the results of a kinematic analysis of the Galactic proper motions to be estimated within the Ogorodnikov—Milne model, we have shown that the slope of the Galactic rotation curve and the Oort parameter C are most sensitive to the transition from the Tycho-2 frame to the TGAS one. Their relative changes after the transformation to the TGAS frame reach 56 and 100%, respectively. At the same time, the changes in the estimates of the Oort parameters A and B as well as the linear velocity of the Sun relative to the Galactic center, the Galactic rotation period, the ratio of the epicyclic frequency to the angular velocity of Galactic rotation, and the mass of the Galaxy within the Galactocentric distance of the Sun are not so large, being 2–10%.

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1. INTRODUCTION

This paper is devoted to an investigation of the Tycho-2 catalogue (Hoeg et al. 2000) based on Gaia Data Release 1 (Lindegren et al. 2016).

Two star catalogues appeared as a result of the Hipparcos space project (ESA 1997). The first of them is Hipparcos. It contains the positions, proper motions, and parallaxes for 118 218 stars measured with an error of ∼1 milliarcsecond (mas). The second catalogue was named Tycho. It provides slightly less accurate data for 1 058 332 stars. The common characteristics of the two catalogues are as follows: the mean epoch of observations is J1991.25, the frame of the catalogues is the ICRS, the accuracy with which they are tied to the ICRS (along three axes) is ±0.6 mas, and the residual rotation relative to the ICRF (along three axes) is ±0.25 mas yr^{-1}.

The Tycho-2 catalogue (Hoeg et al. 2000) is an outgrowth of the Tycho project. It is based the measurements performed onboard the Hipparcos spacecraft. More accurate reduction methods were used in its production, while ground-based catalogues of stellar positions, which gave a large epoch difference, were used to calculate the proper motions. The Astrographic Catalogue (Urban et al. 1998) was used as the first epoch (1905). The second epoch, the mean epoch of Hipparcos observations, is 1991.25.

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In addition to the Astrographic Catalogue, 143 more ground-based catalogues were used. According to the authors, Tycho-2 has a positional accuracy from 10 to 100 mas, depending on the magnitude of a star. The accuracy of proper motions is 2.5 mas yr^{-1}. The 99% completeness of the catalogue is achieved for stars with $V = 11$. The Tycho-2 frame coincides with the Hipparcos one. Being one of the most accurate astrometric catalogues covering the entire celestial sphere, Tycho-2 is widely used in various astrometric problems. Nevertheless, a major shortcoming of this catalogue is the absence of stellar parallaxes. The Tycho-2 Spectral Type catalogue (Wright et al. 2003) provided a partial compensation for this shortcoming. It gives (mostly for stars in the southern equatorial hemisphere) information about the spectra and spectral types of stars that allows the photometric parallaxes to be estimated (Popov et al. 2006).

In Gaia Data Release 1 (Gaia DR1) (Lindegren et al. 2016) the stellar positions and parallaxes were obtained for 2 057 050 Tycho-2 stars at a submilliarcsecond accuracy level. The accuracy of stellar proper motions turned out to be lower: the standard errors in the stellar proper motions reach 2–3 mas yr^{-1}. This catalogue was named TGAS (Tycho–Gaia Astrometric Solution). The TGAS stellar proper motions were obtained by the method of two epochs. The observations onboard the Hipparcos satellite were taken as the first epoch (1991.25), while the second epoch (2015.0) corresponds to the observations onboard the Gaia spacecraft in the first 14 months of its active operation. The systematic errors of the TGAS stars, which depend on the positions on the sphere and colors of stars, are at a level of ±0.3 mas. The stellar positions and proper motions were referred to the reference frame brought into coincidence with ICRF2 at epoch J2015.0 with an accuracy better than 0.1 mas and a residual rotation up to 0.03 mas yr^{-1}. The reference frames of the Hipparcos and Tycho-2 catalogues were found to rotate relative to the Gaia DR1 frame with an angular velocity of 0.24 mas yr^{-1} (Lindegren et al. 2016). The TGAS catalogue may be considered to be the third version of the Tycho project. Compared to Tycho-2, it contains the stellar parallaxes; in addition, its proper motions obtained without invoking any ground-based observations are based on the observations performed in space onboard the Hipparcos and Gaia spacecraft.

The first attempt to compare the TGAS and Tycho-2 catalogues was made by Lindegren et al. (2016). The stellar proper motions of the Tycho-2 catalogue were first transformed to the Gaia DR1 frame using the following transformations:

$$\mu_\alpha^* = (\mu_\alpha^*)_{\text{Tycho-2}} - 0.126\sin\delta\cos\alpha + 0.185\sin\delta\sin\alpha - 0.075\cos\delta, \quad (1)$$

$$\mu_\delta = (\mu_\delta)_{\text{Tycho-2}} - 0.126\sin\alpha - 0.185\cos\alpha. \quad (2)$$

Thereafter, the global statistics, the median (med) and robust scatter estimate (σ) of the proper motion differences TGAS−Tycho-2 in mas yr^{-1} were calculated irrespective of the stellar positions, colors, and other characteristics:

$$\text{med}(\Delta\mu_\alpha^*) = +0.07\ \text{mas}; \quad (3)$$
$$\text{med}(\Delta\mu_\delta) = +0.20\ \text{mas},$$

$$\sigma(\Delta\mu_\alpha^*) = 3.6\ \text{mas yr}^{-1}; \quad (4)$$
$$\sigma(\Delta\mu_\delta) = 3.30\ \text{mas yr}^{-1}.$$

Furthermore, maps of the distribution of these differences on the celestial sphere were constructed (Lindegren et al. 2016, Fig. C.7). Since the TGAS and Tycho-2 catalogues were compared to check and validate the results of the Gaia project, the authors found no sufficient grounds to use the Tycho-2 catalogue as a reliable means for validating the Gaia measurements.

Our paper is also devoted to comparing the TGAS and Tycho-2 catalogues, but we solve other problems as well. The first of them is a detailed study of the systematic differences between the stellar positions and proper motions of these catalogues to transform the measurements performed in the Tycho-2 frame to the TGAS one. The statement of this problem is dictated by the intrinsic logic of astrometry: each new catalogue is always related to the previous catalogues by the systematic differences between the positions and proper motions of their common stars. The second problem is to study the mutual orientation and mutual rotation of the Tycho-2 and TGAS reference frames. The third problem is devoted to searching for the Galactic rotation parameters whose estimates are most sensitive to the transition from the Tycho-2 frame to the TGAS one.

2. INITIAL DATA

To compare the TGAS and Tycho-2 catalogues, we calculated the following individual differences at epoch J2000.0 of Tycho-2: $\Delta\alpha^* = (\alpha_{\text{TGAS}} - \alpha_{\text{Tycho-2}})\cos\delta$, $\Delta\delta = (\delta_{\text{TGAS}} - \delta_{\text{Tycho-2}})$, $\Delta\mu_\alpha^* = ((\mu_\alpha)_{\text{TGAS}} - (\mu_\alpha)_{\text{Tycho-2}})\cos\delta$, $\Delta\mu_\delta = ((\mu_\delta)_{\text{TGAS}} - (\mu_\delta)_{\text{Tycho-2}})$. The TGAS stellar positions were transformed to epoch J2000.0 using a linear procedure. We did not apply the rigorous procedure (ESA 1997), first, because of the absence of radial velocities for all stars and, second, because of the presence of unrealistically large differences for a large number of stars, as can be seen from Figs. 1 and 2. For this

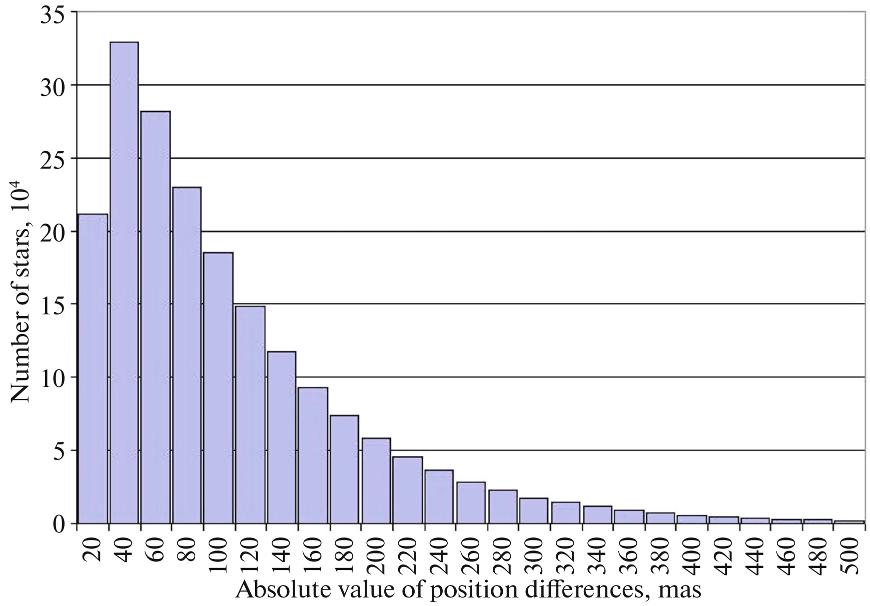

Fig. 1. (Color online) Histogram of the absolute values $\sqrt{(\Delta\alpha^*)^2 + (\Delta\delta)^2}$ of the J2000 position differences.

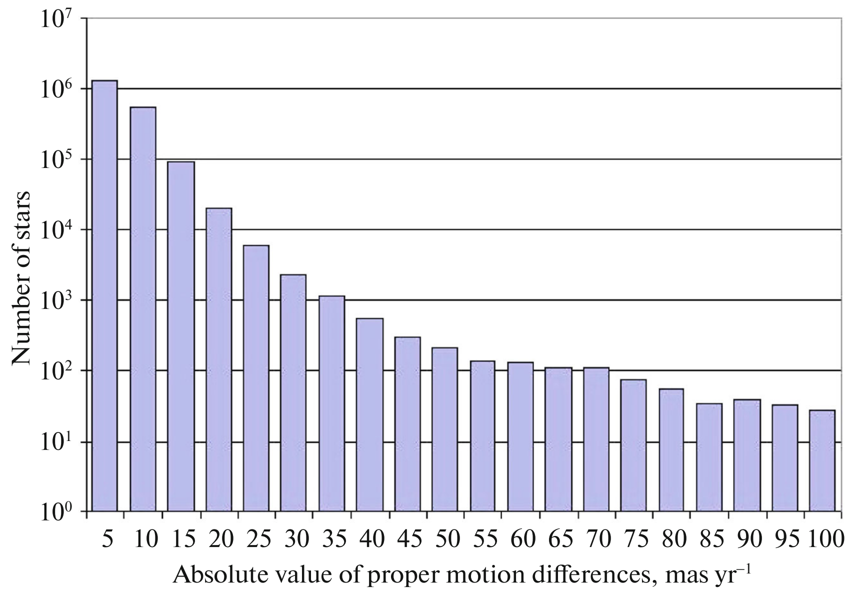

Fig. 2. (Color online) Histogram of the absolute values $\sqrt{(\Delta\mu_\alpha^*)^2 + (\Delta\mu_\delta)^2}$ of the stellar proper motion differences.

reason, we filtered the data by retaining the stars for which the following relations held:

$$\sqrt{(\Delta\alpha^*)^2 + (\Delta\delta)^2} < A, \quad (5)$$

$$\sqrt{(\Delta\mu_\alpha^*)^2 + (\Delta\mu_\delta)^2} < M, \quad (6)$$

where $A = 200$ mas and $M = 20$ mas yr^{-1}. The remaining differences were divided into three groups for the stars in the magnitude ranges $G_1 = 10{-}11$, $G_2 = 11{-}12$, and $G_3 = 12{-}14$ and were referred to 1200 pixels of the celestial sphere constructed by the Healpix method (Gorski et al. 2005). For each pixel we calculated the mean differences $\langle\Delta\alpha^*\rangle$, $\langle\Delta\delta\rangle$, $\langle\Delta\mu_\alpha^*\rangle$, $\langle\Delta\mu_\delta\rangle$ and the root-mean-square (rms) deviations of the individual differences from the means. Obviously, these characteristics determine the mean deviations and dispersions of the Tycho-2 stellar positions and proper motions from the TGAS ones in a given direction on the celestial sphere and in a given magnitude range. If the TGAS accuracy is assumed to be considerably higher than the Tycho-2

Table 1. Global statistics of the stellar position and proper motion differences TGA−Tycho-2

G	10−11	11−12	12−14
number of stars in sample	538323	648279	126755
mean($\langle\Delta\mu^*_\alpha\rangle$) mas yr^{-1}	0.15	0.14	0.14
med($\langle\Delta\mu^*_\alpha\rangle$), mas yr^{-1}	0.13	0.17	0.09
$\sigma(\langle\Delta\mu^*_\alpha\rangle)$, mas yr^{-1}	3.41	3.89	4.18
mean($\langle\Delta\mu_\delta\rangle$), mas yr^{-1}	0.27	0.40	0.48
med($\langle\Delta\mu_\delta\rangle$), mas yr^{-1}	0.26	0.44	0.54
$\sigma(\langle\Delta\mu_\delta\rangle)$, mas yr^{-1}	3.12	3.48	3.79
mean($\langle\Delta\alpha^*\rangle$), mas	1.30	2.02	2.23
med($\langle\Delta\alpha^*\rangle$), mas	1.40	1.68	1.51
$\sigma(\langle\Delta\alpha^*\rangle)$, mas	53.98	76.33	86.43
mean($\langle\Delta\delta\rangle$), mas	1.78	4.36	4.69
med($\langle\Delta\delta\rangle$), mas	2.74	4.34	3.57
$\sigma(\langle\Delta\delta\rangle)$, mas	54.59	75.97	86.21

one, then these quantities may be called the Tycho-2 errors. In turn, averaging these quantities over all pixels gives an idea of the mean errors and dispersions of the Tycho-2 positions and proper motions relative to TGAS over the entire sky, but in the former magnitude ranges. These global statistics of the Tycho-2 catalogue are given in Table 1, whose analysis allows the following conclusion to be formulated: all global statistics of the stellar proper motion and position differences between the TGAS and Tycho-2 catalogues increase when passing from bright stars to fain ones (except for the differences $\Delta\mu^*_\alpha$). It should be noted that our results agree with the results (3) and (4). However, when comparing them, it should be kept in mind that we did not reduce Tycho-2 to the TGAS frame of the form (1) and (2), because within the formulated problem of the data transformation from the Tycho-2 frame to the TGAS one these rotations must be contained in the numerical coefficients of the model of systematic differences, which, obviously, will depend on the magnitude of stars. Hence it follows that the Tycho-2 positions and proper motions are distorted by the errors of the magnitude equation. The latter appear, because old photographic catalogues were used in constructing Tycho-2. This conclusion necessitates using an appropriate mathematical apparatus in calculating the systematic differences to be discussed in the next section.

3. REPRESENTATION OF THE SYSTEMATIC DIFFERENCES TGAS−Tycho-2 BY VECTOR SPHERICAL HARMONICS

In this paper the systematic differences between the TGAS and Tycho-2 catalogues were obtained by the method based on the application of vector spherical harmonics including the magnitude equation that we developed previously (Vityazev and Tsvetkov 2015a). Below we describe its main steps to obtain the systematic position and proper motion differences TGAS−Tycho-2. In this method the vector fields of right ascension and declination differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ as well as the fields of stellar proper motion differences $\Delta\mu_\alpha\cos\delta$ and $\Delta\mu_\delta$

$$\Delta\mathbf{F}(\alpha,\delta,m)=\begin{cases}\Delta\alpha\cos\delta\mathbf{e}_\alpha+\Delta\delta\mathbf{e}_\delta\\ \Delta\mu_\alpha\cos b\mathbf{e}_\alpha+\Delta\mu_\delta\mathbf{e}_\delta\end{cases}\tag{7}$$

are approximated by the expression

$$\begin{aligned}&\Delta\mathbf{F}(\alpha,\delta,G)\\&=\sum_{nkpr}t_{nkpr}\mathbf{T}_{nkp}(\alpha,\delta)Q_r(\bar{G})\\&+\sum_{nkpr}s_{nkpr}\mathbf{S}_{nkp}(\alpha,\delta)Q_r(\bar{G}).\end{aligned}\tag{8}$$

In Eq. (7) the mutually orthogonal unit vectors in the directions of change of the right ascensions and declinations in the tangential plane are denoted by $\mathbf{e}_\alpha$ and $\mathbf{e}_\delta$, respectively. In Eq. (8) the dependence of the fields on coordinates is described by toroidal, $\mathbf{T}_{nkp}(\alpha,\delta)$, and spheroidal, $\mathbf{S}_{nkp}(\alpha,\delta)$, spherical harmonics, while the dependence of the field on magnitude G is described by normalized Legendre polynomials $Q_r(\bar{G})$. Explicit formulas to calculate the functions $\mathbf{T}_{nkp}(\alpha,\delta)$, $\mathbf{S}_{nkp}(\alpha,\delta)$, and $Q_r(\bar{G})$ are provided in the Appendix. A detailed description of the procedure for calculating the coefficients t_{nkpr} and s_{nkpr} is contained in Vityazev and Tsvetkov (2015a, 2015b). Note that in comparison with the traditional techniques of using vector spherical harmonics (Mignard and Morando 1990; Mignard and Froeschle 2000; Mignard and Klioner 2012), new features of our method are, first, the method of determining the significance of all the harmonics that can be realized on a chosen data pixelization scheme and, second, modeling the magnitude equation and determining its parameters.

The coefficients t_{nkpr} and s_{nkpr} were obtained in two steps. In the first step, we calculated the coefficients $t_{nkp}(G_j)$ and $s_{nkp}(G_j)$ for each of our three samples of stars with means $G_1=10.5$, $G_2=11.5$, and $G_3=13$. Since the vector spherical harmonics are orthonormal, the rms errors of the

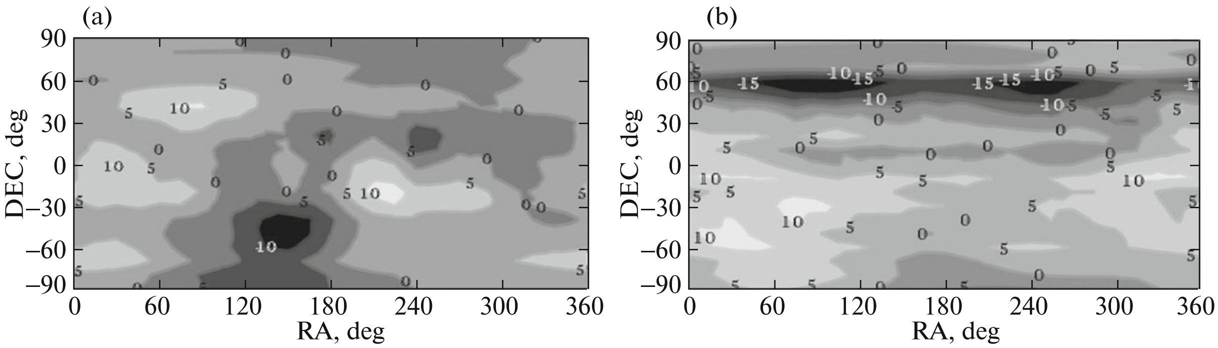

Fig. 3. Maps of systematic position differences TGAS−Tycho-2 for $G_1 = 10{-}11$. (a) $\Delta\alpha\cos\delta$ and (b) $\Delta\delta$. The units are mas.

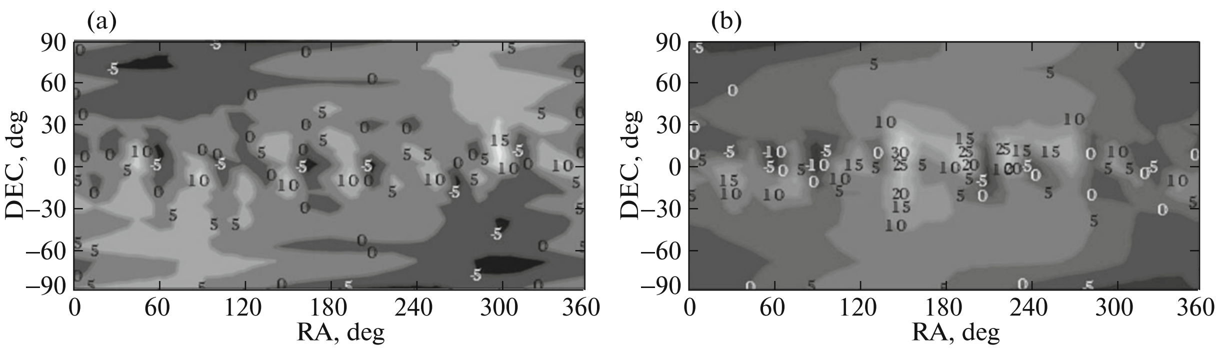

Fig. 4. Maps of systematic position differences TGAS−Tycho-2 for $G_3 = 12{-}14$. (a) $\Delta\alpha\cos\delta$ and (b) $\Delta\delta$. The units are mas.

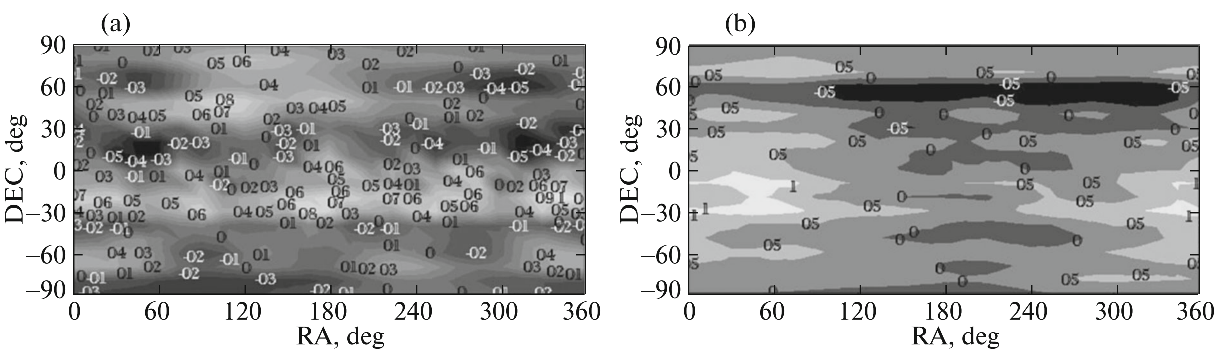

Fig. 5. Maps of systematic proper motion differences TGAS−Tycho-2 for $G_1 = 10{-}11$. (a) $\Delta\mu_\alpha\cos\delta$ and (b) $\Delta\mu_\delta$. The units are mas yr^{-1}.

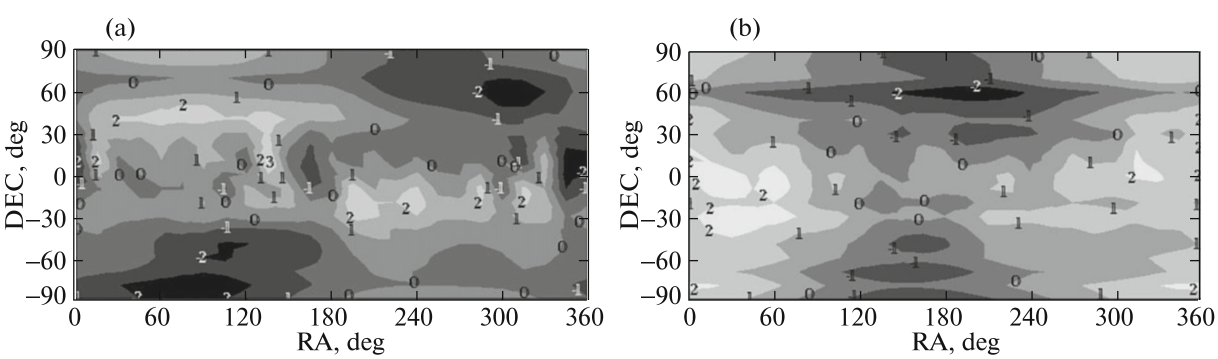

Fig. 6. Maps of systematic proper motion differences TGAS−Tycho-2 for $G_3 = 12{-}14$. (a) $\Delta\mu_\alpha\cos\delta$ and (b) $\Delta\mu_\delta$. The units are mas yr^{-1}.

Table 2. Ranges of variations in the systematic differences of the right ascensions and proper motions in right ascension TGAS−Tycho-2

G	$\min(\Delta\alpha\cos\delta)$	$\max(\Delta\alpha\cos\delta)$	$\min(\Delta\mu_\alpha\cos\delta)$	$\max(\Delta\mu_\alpha\cos\delta)$
10−11	−13.5	13.6	−0.94	1.17
11−12	−11.0	16.5	−1.30	1.55
12−14	−15.4	19.8	−3.45	3.28

The units are: mas (columns 2, 3) and mas yr^{-1} (columns 4, 5).

toroidal and spheroidal coefficients $\sigma_t(G)$ and $\sigma_s(G)$ in each sample turned out to be identical and equal to $\sigma_t(G_1) = \sigma_s(G_1) = 0.60$, $\sigma_t(G_2) = \sigma_s(G_2) = 0.67$, and $\sigma_t(G_3) = \sigma_s(G_3) = 1.74$ mas for the position differences and $\sigma_t(G_1) = \sigma_s(G_1) = 0.05$, $\sigma_t(G_2) = \sigma_s(G_2) = 0.06$, and $\sigma_t(G_3) = \sigma_s(G_3) = 0.12$ mas yr^{-1} for the stellar proper motion differences, respectively. The dependence of each of the coefficients on magnitude was approximated by a parabola (the sum of Legendre polynomials $Q_r(\bar{G})$, $r = 0, 1, 2$) with coefficients t_{nkpr} and s_{nkpr}. The parabolic representation of the coefficients was made in the case where at least one of the coefficients in any sample in G was significant (in this case, the like insignificant coefficients in other zones were assumed to be zero). The derived coefficients t_{nkpr} and s_{nkpr} are presented in Tables 13−18. Their significance was established with a probability of 97.7−99.9% using the statistical criteria described in Vityazev and Tsvetkov (2015a, 2015b).

For the position differences the rms errors of these coefficients are $\sigma(t_{nkp0}) = \sigma(s_{nkp0}) = 1.18$, $\sigma(t_{nkp1}) = \sigma(s_{nkp1}) = 0.75$, and $\sigma(t_{nkp2}) = \sigma(s_{nkp2}) = 0.45$ mas, while for the proper motions $\sigma(t_{nkp0}) = \sigma(s_{nkp0}) = 0.07$, $\sigma(t_{nkp1}) = \sigma(s_{nkp1}) = 0.05$, and $\sigma(t_{nkp2}) = \sigma(s_{nkp2}) = 0.04$ mas yr^{-1} for all indices nkp. Because of this, the tables give the ratios of the absolute values of the coefficients to their rms errors.

Table 3. Ranges of variations in the systematic differences of the declinations and proper motions in declination TGAS−Tycho-2

G	$\min(\Delta\delta)$	$\max(\Delta\delta)$	$\min(\Delta\mu_\delta)$	$\max(\Delta\mu_\delta)$
10−11	−21.1	12.2	−1.01	1.33
11−12	−14.9	18.8	−2.17	2.10
12−14	−17.1	33.7	−2.55	2.79

The units are: mas (columns 2, 3) and mas yr^{-1} (columns 4, 5).

Formula (8) and the derived coefficients t_{nkpr} and s_{nkpr} solve the formulated problem, because they allow the systematic stellar position and proper motion differences to be calculated for the transition from the Tycho-2 frame to the TGAS one and back. Figures 3−6 give an idea of how the systematic differences vary over the celestial sphere. The ranges of systematic difference variations are shown in Tables 2 and 3.

The maps of systematic position and proper motion differences TGAS−Tycho-2 exhibit two characteristic features: the zonal structure of the systematic differences and their change when passing to faint stars (magnitude equation). Obviously, both these effects are a consequence of using the zonal photographic catalogues that provide a basis for the Astrographic Catalogue to construct the Tycho-2 proper motions.

The effect of the magnitude equation is noticed when comparing the maps of systematic differences for various groups of stars in the G_1 and G_3 bands (Figs. 3−6). Figures 7 and 8 demonstrate the quantitative characteristics of these effects. The differences of $\Delta\alpha\cos\delta$, $\Delta\delta$, $\Delta\mu_\alpha\cos\delta$, and $\Delta\mu_\delta$ calculated for the stars of the first ($G_1 = 10{-}11$) and third ($G_3 = 12{-}14$) samples are shown here. The ranges of variations in these quantities are given in Tables 4 and 5.

4. THE MODEL OF SYSTEMATIC STELLAR POSITION AND PROPER MOTION DIFFERENCES

The main purpose of the systematic differences is to transform the stellar positions and proper motions of one catalogue to the frame of another catalogue. However, apart from these reduction tasks, the systematic position and proper motion differences for the same stars reflect the discrepancies between the reference frames that are realized by the comparison catalogues.

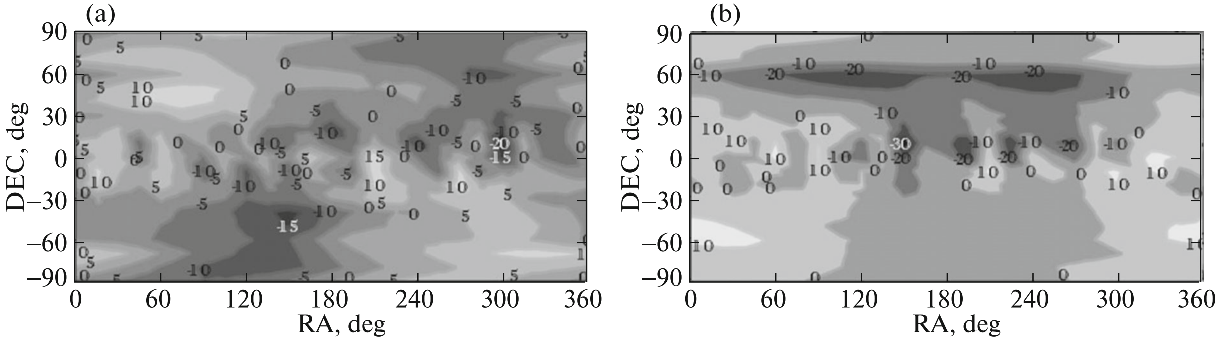

Fig. 7. Maps of variations in the systematic stellar position differences TGAS−Tycho-2 when passing from the G_1 photometric band to G_3. (a) $\Delta\alpha\cos\delta$ and (b) $\Delta\delta$. The units are mas.

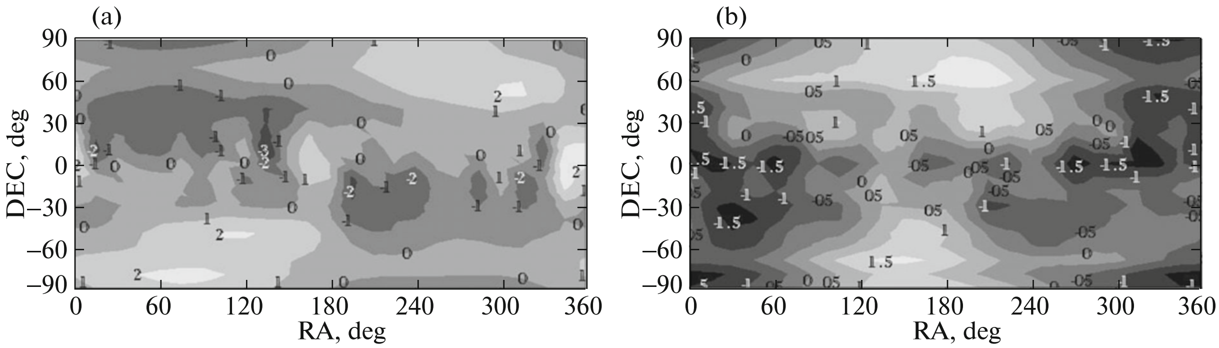

Fig. 8. Maps of variations in the systematic stellar proper motion differences TGAS−Tycho-2 when passing from the G_1 photometric band to G_3. (a) $\Delta\mu_\alpha\cos\delta$ and (b) $\Delta\mu_\delta$. The units are mas yr^{-1}.

The model of rigid-body mutual rotation and displacement of two reference frames is commonly used in comparing optical catalogues. We will assume that the coordinate system (x, y, z) of catalogue Cat is transformed to the coordinate system (x', y', z') of catalogue Cat′ by the translation of the coordinate origin to point (ξ, η, ζ) and the rotation of the axes through angles (Ω_1, Ω_2, Ω_3). We will assume the angles to be positive when the rotation for an observer located on the positive ray of the axis is counterclockwise when viewed along the rotation axis toward the coordinate origin. In this case, the transformation of the coordinate systems is described by the following equation:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M \begin{bmatrix} x-\xi \\ y-\eta \\ z-\zeta \end{bmatrix}. \qquad (9)$$

In the approximation of an infinitesimal rotation the matrix M is

$$M = \begin{bmatrix} 1 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 1 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 1 \end{bmatrix}. \qquad (10)$$

Under the analogous assumption of a small displacement of the coordinate origin, to within terms of the

Table 4. Differences $\Delta A = (\Delta\alpha\cos\delta)(G_1) - (\Delta\alpha\cos\delta)(G_3)$, and $\Delta D = (\Delta\delta)(G_1) - (\Delta\delta)(G_3)$

min(ΔA)	max(Δ)A	min(ΔD)	max(ΔD)
−22.0	15.2	−35.8	14.8

The units are: mas (columns 2, 3) and mas yr^{-1} (columns 4, 5).

Table 5. Differences $\Delta PMA = (\Delta\mu_{\alpha)}\cos\delta)(G_1) - (\Delta\mu_\alpha\cos\delta)(G_3)$, and $\Delta PMD = (\Delta\mu_{\delta)})(G_1) - (\Delta\mu_\delta)(G_3)$

min(ΔPMA)	max(ΔPMA)	min(ΔPMD)	max(ΔPMD)
−3.3	3.0	−2.0	1.9

The units are: mas (columns 2, 3) and mas yr^{-1} (columns 4, 5).

first order of smallness, we have

$$\begin{bmatrix} x'-x \\ y'-y \\ z'-z \end{bmatrix} = (M-E)\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \quad (11)$$

where E is a unit matrix.

Using the relations between the rectangular and spherical coordinates of a star with a heliocentric distance r

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\delta\cos\alpha \\ r\cos\delta\sin\alpha \\ r\sin\delta \end{bmatrix}, \quad (12)$$

we obtain

$$\begin{bmatrix} r\Delta\alpha\cos\delta \\ r\Delta\delta \\ \Delta r \end{bmatrix} = A\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}, \quad (13)$$

where A is the transformation matrix of the unit vectors of the rectangular coordinate system to the unit vectors of the tangent plane directed toward increasing right ascensions, declinations, and radius vector of the star:

$$A = \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha\sin\delta & -\sin\alpha\sin\delta & \cos\delta \\ \cos\alpha\cos\delta & \sin\alpha\cos\delta & \sin\delta \end{bmatrix}. \quad (14)$$

Assuming the displacement and rotation parameters to be functions of time,

$$\xi(t) = \xi_0 + \dot{\xi}(t-t_0), \quad (15)$$
$$\eta(t) = \eta_0 + \dot{\eta}(t-t_0),$$
$$\xi(t) = \zeta_0 + \dot{\zeta}(t-t_0),$$

$$\Omega_1(t) = \varepsilon_1 + \omega_1(t-t_0), \quad (16)$$
$$\Omega_2(t) = \varepsilon_2 + \omega_2(t-t_0),$$
$$\Omega_3(t) = \varepsilon_3 + \omega_3(t-t_0),$$

from (11)–(13) at $\Delta x = x'-x$, $\Delta y = y'-y$, and $\Delta z = z'-z$ we obtain

$$\Delta\alpha\cos\delta = \varepsilon_1\sin\delta\cos\alpha + \varepsilon_2\sin\delta\sin\alpha - \varepsilon_3\cos\delta + g_1\sin\alpha - g_2\cos\alpha; \quad (17)$$

$$\Delta\delta = -\varepsilon_1\sin\alpha + \varepsilon_2\cos\alpha + g_1\cos\alpha\sin\delta + g_2\sin\alpha\sin\delta - g_3\cos\delta; \quad (18)$$

$$\Delta r = -\xi\cos\alpha\cos\delta - \eta\sin\alpha\cos\delta - \zeta\sin\delta; \quad (19)$$

$$\Delta\mu_\alpha\cos\delta = \omega_1\sin\delta\cos\alpha + \omega_2\sin\delta\sin\alpha - \omega_3\cos\delta + \dot{g}_1\sin\alpha - \dot{g}_2\cos\alpha; \quad (20)$$

$$\Delta\mu_\delta = -\omega_1\sin\alpha + \omega_2\cos\alpha + \dot{g}_1\cos\alpha\sin\delta + \dot{g}_2\sin\alpha\sin\delta - \dot{g}_3\cos\delta; \quad (21)$$

$$\Delta V_r = -\dot{\xi}\cos\alpha\cos\delta - \dot{\eta}\sin\alpha\cos\delta - \dot{\zeta}\cos\delta. \quad (22)$$

The parameters in these equations have the following meaning:

- $\Delta\alpha$, $\Delta\delta$, $\Delta\mu_\alpha$, $\Delta\mu_\delta$ are the stellar position and proper motion differences Cat′−Cat at epoch t_0;
- Δr and ΔV_r are the change in the heliocentric distance to the star expressed in units of distance and the change in the radial velocity of the star expressed in km s^{-1};
- ε_1, ε_2, ε_3 are the rotation angles of the rectangular coordinate system of Cat relative to Cat′ expressed in mas;
- ω_1, ω_2, ω_3 are the angular velocities of the rectangular coordinate system of Cat relative to Cat′ expressed in mas yr^{-1};
- $g_1 = \xi_0/r$, $g_2 = \eta_0/r$, $g_3 = \zeta_0/r$ are the components of the displacement of the zero point of the Cat reference frame relative to the Cat′ reference frame expressed in mas;
- $\dot{g}_1 = \dot{\xi}/r$, $\dot{g}_2 = \dot{\eta}/r$, $\dot{g}_3 = \dot{\zeta}/r$ are the components of the displacement velocity of the zero point of the Cat reference frame relative to the Cat′ reference frame expressed in mas yr^{-1}.

Note that for the vector $\Delta = \Delta\alpha\cos\delta\mathbf{e}_\alpha + \Delta_\delta\mathbf{e}_\delta$ and the component Δr Eqs. (17)–(19) can be written as a linear combination of vector spherical harmonics:

$$\Delta = -\varepsilon_1\frac{\mathbf{T}_{111}(\alpha,\delta)}{\rho_{11}} - \varepsilon_2\frac{\mathbf{T}_{110}(\alpha,\delta)}{\rho_{11}} - \varepsilon_3\frac{\mathbf{T}_{101}(\alpha,\delta)}{\rho_{10}} - g_1\frac{\mathbf{S}_{111}(\alpha,\delta)}{\rho_{11}} - g_2\frac{\mathbf{S}_{110}(\alpha,\delta)}{\rho_{11}} - g_3\frac{\mathbf{S}_{101}(\alpha,\delta)}{\rho_{10}}, \quad (23)$$

$$\Delta r = -\xi\frac{K_{111}(\alpha,\delta)}{R_{11}} - \eta\frac{K_{110}(\alpha,\delta)}{R_{11}} - \zeta\frac{K_{101}(\alpha,\delta)}{R_{10}}. \quad (24)$$

Obviously, an analogous representation of Eqs. (20)–(22) as a linear combination of vector spherical harmonics can also be written for the vector $\Delta_\mu = \Delta\mu_\alpha \cos\delta \mathbf{e}_\alpha + \Delta\mu_\delta \mathbf{e}_\delta$ and the component ΔV_r:

$$\Delta_\mu = -\omega_1 \frac{\mathbf{T}_{111}(\alpha,\delta)}{\rho_{11}} - \omega_2 \frac{\mathbf{T}_{110}(\alpha,\delta)}{\rho_{11}} - \omega_3 \frac{\mathbf{T}_{101}(\alpha,\delta)}{\rho_{10}} - \dot{g}_1 \frac{\mathbf{S}_{111}(\alpha,\delta)}{\rho_{11}} - \dot{g}_2 \frac{\mathbf{S}_{110}(\alpha,\delta)}{\rho_{11}} - \dot{g}_3 \frac{\mathbf{S}_{101}(\alpha,\delta)}{\rho_{10}}, \quad (25)$$

$$\Delta V_r = -\dot{\xi} \frac{K_{111}(\alpha,\delta)}{R_{11}} - \dot{\eta} \frac{K_{110}(\alpha,\delta)}{R_{11}} - \dot{\zeta} \frac{K_{101}(\alpha,\delta)}{R_{10}}. \quad (26)$$

In these equations

$$\rho_{nk} = \frac{R_{nk}}{\sqrt{n(n+1)}}, \quad (27)$$

while R_{nk} is calculated from Eq. (43).

These are the basic equations for analyzing the discrepancies between the two reference frames due to their mutual displacement and rotation. In our case, the Tycho-2 and TGAS catalogues act as Cat and Cat′, respectively. Since these catalogues contain no radial velocities, Eqs. (19), (24) and (22), (26) will not be used in our analysis. Here, they are given to keep the generality of our reasoning.

As we see, the model of the systematic differences that are generated by the mutual rotation and displacement of the reference frames is a linear combination of first-order toroidal and spheroidal harmonics. Comparing Eqs. (17)–(18) and (20)–(22) with Eqs. (23)–(25) term by term, we conclude that the rotation and displacement parameters can be calculated via the coefficients of the decomposition of systematic differences into vector spherical harmonics, as shown in Table 6. Note that the same relations can be derived by directly decomposing the right-hand sides of Eqs. (17)–(19) and (20)–(22) into a system of vector spherical harmonics.

The relationship between the rotation angles of one coordinate system relative to another and the coefficients of the decomposition of the systematic differences between the right ascensions and declinations of stars into scalar harmonics was established within the model of rigid-body rotation by Vityazev (1994). Such a relationship was found by Mignard and Morando (1990) when using vector spherical harmonics. The influence of the displacement of the zero points of the coordinate systems corresponding to two comparison catalogues on the systematic differences was revealed in Mignard and Klioner (2012), where this effect was considered as a regular flow (a glide) from one point to another on the sphere diametrically opposed. From an astronomical point of view, this field can be associated with the observer's motion toward the apex, whereby the systematic differences "outflow" from the apex and "inflow" into the antiapex.

In Fig. 9 on the left we see the vector field due to the rotations of the Tycho-2 reference frame through angles $\varepsilon_1 = -t_{111}/2.89$, $\varepsilon_2 = -t_{110}/2.89$, and $\varepsilon_3 = -t_{101}/2.89$. The values of t_{nkp} here were calculated for the sample $G_2 = 11{-}12$ from the formula

$$t_{nkp}(G_2) = \sum_r t_{nkpr} Q_r(\bar{G}_2), \quad (28)$$

with the coefficients t_{nkpr} taken from Table 15.

The coordinates of the rotation pole corresponding to the end of the vector with these components were derived from the formulas

$$A_{\rm rot} = \arctan\left(\frac{\varepsilon_2}{\varepsilon_1}\right); \quad (29)$$

$$D_{\rm rot} = \arctan\left(\frac{\varepsilon_3}{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}\right).$$

The vector field of systematic position differences TGAS–Tycho-2 generated by the displacement of the zero points of these reference frames by $g_1 = -s_{111}/2.89$, $g_2 = -s_{110}/2.89$, and $g_3 = -s_{101}/2.89$ is shown in Fig. 9 on the right. The values of s_{nkp} here were calculated for the sample $G_2 = 11{-}12$ from the formula

$$s_{nkp}(G_2) = \sum_r s_{nkpr} Q_r(\bar{G}_2), \quad (30)$$

with the coefficients t_{nkpr} taken from Table 13.

Table 6. Coefficients of the vector spherical harmonic decomposition of Eqs. (17), (18) and (20), (21)

Coefficient	Eqs. (17), (18)	Eqs. (20), (21)
t_{101}	$-2.89\varepsilon_3$	$-2.89\omega_3$
t_{110}	$-2.89\varepsilon_2$	$-2.89\omega_2$
t_{111}	$-2.89\varepsilon_1$	$-2.89\omega_1$
s_{101}	$-2.89g_3$	$-2.89\dot{g}_3$
s_{110}	$-2.89g_2$	$-2.89\dot{g}_2$
s_{111}	$-2.89g_1$	$-2.89\dot{g}_1$

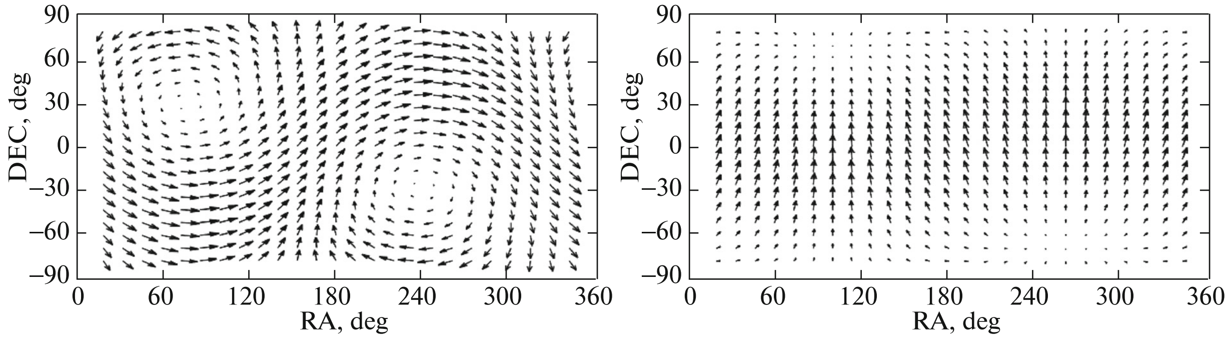

Fig. 9. Vector maps of position differences TGAS−Tycho-2 for $G = 10{-}11$ due to the rotation of the Tycho-2 reference frame relative to the TGAS one (left) and the displacement of their zero points (right).

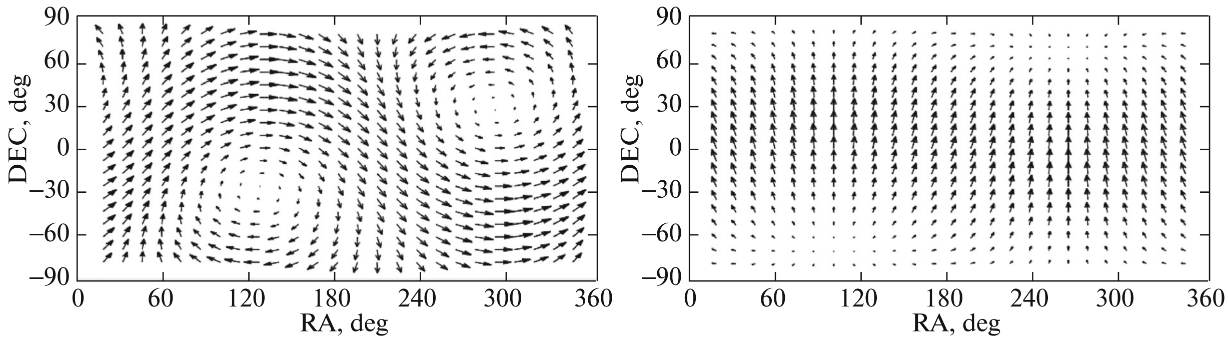

Fig. 10. Vector maps of stellar proper motion differences TGAS−Tycho-2 for $G = 10{-}11$ due to the rotation of the Tycho-2 reference frame relative to the TGAS one (left) and the displacement velocity of their zero points (right).

The coordinates of the outflow point are determined from the formulas

$$A = \arctan\left(\frac{g_2}{g_1}\right); \qquad (31)$$

$$D = \arctan\left(\frac{g_3}{\sqrt{g_1^2 + g_2^2}}\right).$$

Similarly, in Fig. 10 on the left we see the vector field due to the rotation of the Tycho-2 reference frame relative to the TGAS one. The right panel of the same figure shows the vector field generated by the displacement velocity of the zero points of our reference frames. Tables 18, 16 and Eqs. (28), (29) with a formal substitution of $\dot{g}_i$ for g_i and ω_i for ε_i were used to construct these fields.

The derived orientation and rotation parameters of the Tycho-2 reference frame relative to the TGAS one are given in Tables 7 and 8. Analysis of these tables shows that the orientation and rotation parameters of the Tycho-2 frame depend on the magnitude of stars and increase as we pass to fainter stars. Therefore, it should be noted that Lindegren et al. (2016) used the rotation parameters $\omega_x = -0.126$, $\omega_y = -0.185$, and $\omega_z = 0.076$ derived from the stellar proper motion differences between the TGAS and Hipparcos catalogues when transforming the Tycho-2 stellar proper motions to the TGAS frame (Gaia DR1). The fact that, contrary to the analogous parameters of the Hipparcos frame, the rotation parameters of the Tycho-2 reference frame relative to the TGAS one turned out to be dependent on the magnitude of stars is quite understandable, because ground-based catalogues were used to obtain the Tycho-2 stellar proper motions and were not used to obtain the Hipparcos stellar proper motions.

4.1. The Influence of Gaia on the Estimates of Stellar Velocity Field Parameters

Let us now use our model of the systematic differences between the equatorial proper motions of stars to estimate the changes in the kinematic parameters of the stellar proper motions when passing from the Tycho-2 frame to the TGAS one.

In the Ogorodnikov–Milne model (Ogorodnikov 1965; du Mont 1977), which is often used in analyzing the stellar proper motions, the stellar velocity field is represented by a linear relation:

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{\Omega} \times \mathbf{r} + \mathbf{M}^{+}\mathbf{r}, \qquad (32)$$

where $\mathbf{V}$ is the stellar velocity, $\mathbf{V}_0$ is the influence of the translational solar motion, $\mathbf{\Omega}$ is the angular

Table 7. Orientation parameters and rotation angle $\varepsilon = \sqrt{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2}$ of the Tycho-2 reference frame relative to the TGAS one

G	ε_x	ε_y	ε_z	ε	$A_{\rm rot}$	$D_{\rm rot}$
10–11	-1.20 ± 0.22	2.79 ± 0.22	-1.75 ± 0.22	3.51 ± 0.22	113.3 ± 4.1	-29.9 ± 3.5
11–12	-1.56 ± 0.23	-3.48 ± 0.23	-2.36 ± 0.22	4.49 ± 0.23	245.9 ± 3.5	-31.8 ± 3.0
12–14	0.00 ± 0.58	-4.96 ± 0.58	-2.68 ± 0.58	5.63 ± 0.58	270 ± 6.7	-28.4 ± 5.9

The units are: mas (columns 2–5) and degrees (columns 6, 7).

Table 8. Rotation parameters and rotation rate $\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ of the Tycho-2 reference frame relative to the TGAS one

G	ω_x	ω_y	ω_z	ω	$A_{\rm rot}$	$D_{\rm rot}$
10–11	-0.15 ± 0.02	0.26 ± 0.02	-0.19 ± 0.02	0.35 ± 0.02	120.2 ± 3.4	-32.2 ± 2.9
11–12	-0.08 ± 0.02	0.81 ± 0.02	-0.23 ± 0.02	0.85 ± 0.02	95.6 ± 1.6	-15.9 ± 1.6
12–14	0.00 ± 0.04	1.18 ± 0.04	-0.29 ± 0.04	1.22 ± 0.04	90.0 ± 2.1	-14.0 ± 2.0

The units are: mas (columns 2–5) and degrees (columns 6, 7).

velocity of rigid-body rotation of the stellar system, and $\mathbf{M}^+$ is the symmetric deformation tensor of the velocity field.

The Ogorodnikov–Milne model contains 12 parameters:

U, V, W are the components of the translational velocity vector of the Sun $\mathbf{V}_0$ relative to the stars;

ω_1, ω_2, ω_3 are the components of the rigid-body rotation vector $\boldsymbol{\Omega}$;

M^+_{11}, M^+_{22}, M^+_{33} are the parameters of the tensor $\mathbf{M}^+$ describing the velocity field contraction–expansion along the principal Galactic axes;

M^+_{12}, M^+_{13}, M^+_{23} are the parameters of the tensor $\mathbf{M}^+$ describing the velocity field deformation in the principal plane and two planes perpendicular to it.

Projecting Eq. (32) onto the unit vectors of the Galactic coordinate system and using the designation r for the stellar distance and $\mathcal{K} = 4.738$ for the conversion factor of the dimensions of stellar proper motions mas yr^{-1} to km s^{-1} kpc^{-1}, we obtain

$$\mathcal{K}\mu_l \cos b = U/r \sin l - V/r \cos l \quad (33)$$
$$- \omega_1 \sin b \cos l - \omega_2 \sin b \sin l + \omega_3 \cos b$$
$$- M^+_{13} \sin b \sin l + M^+_{23} \sin b \cos l$$
$$+ M^+_{12} \cos b \cos 2l - \frac{1}{2} M^*_{11} \cos b \sin 2l,$$

$$\mathcal{K}\mu_b = U/r \cos l \sin b + V/r \sin l \sin b \quad (34)$$
$$- W/r \cos b + \omega_1 \sin l - \omega_2 \cos l$$
$$- \frac{1}{2} M^+_{12} \sin 2b \sin 2l + M^+_{13} \cos 2b \cos l$$
$$+ M^+_{23} \cos 2b \sin l - \frac{1}{2} M^*_{11} \sin 2b \cos^2 l$$
$$+ \frac{1}{2} M^*_{33} \sin 2b,$$

where $M^*_{11} = M^+_{11} - M^+_{22}$ and $M^*_{33} = M^+_{33} - M^+_{22}$.

It is well known that the Ogorodnikov–Milne model parameters allow a number of kinematic and physical characteristics of our Galaxy to be estimated. First of all, the Oort constants are determined via them. Assuming that the velocity field is axisymmetric or $V_R = 0$, where V_R is the velocity component along the radius vector in the Galactic cylindrical coordinate system (Miyamoto et al. 1993), for the Oort constants A and B we have

$$A = M^+_{12}, \quad B = \Omega_3. \quad (35)$$

In turn, the Oort constants C and K are determined as follows:

$$C = \frac{M^*_{11}}{2}, \quad K = \frac{M^*_{11}}{2} - M^*_{33}. \quad (36)$$

These constants are linear combinations of the gradients of stellar velocity field parameters. The constants A and C characterize the azimuthal and radial deformations, respectively, the constant B characterizes the rigid-body rotation component of the stars relative to the Galactic coordinate system, and K characterizes the field divergence (Torra et al. 2000).

Vityazev and Tsvetkov (2009) performed a decomposition of Eqs. (33) and (34) into vector spherical harmonics (VSHs), which shows what harmonics in the decomposition of the stellar proper motions determine the individual parameters of this model. Since the solar motion components enter into Eqs. (33) and (34) with the factor $1/r$, the coefficients of the corresponding VSHs are determined to within the factor $1/\langle r\rangle$. Obviously, in view of the linearity of the Ogorodnikov–Milne equations, the systematic differences between the stellar proper motions of two catalogues can also be represented as Eqs. (33) and (34), in which the kinematic parameters U/r, V/r, ..., M_{33}^{+} are replaced by the differences $\Delta U/r$, $\Delta V/r$, ..., ΔM_{33}^{+}. Therefore, using the results from Vityazev and Tsvetkov (2009), we can relate the corrections of the kinematic parameters $\Delta U/r$, $\Delta V/r$, ..., ΔM_{33}^{+} when passing from the frame of one catalogue to the frame of another catalogue to the numerical values of the statistically significant decomposition coefficients of the stellar proper motions of these catalogues. The formulas explaining the physical meaning of the decomposition coefficients are given in Table 9.

Table 9. Relations between the differences of the Ogorodnikov–Milne model parameters and the decomposition coefficients of the stellar proper motions of two catalogues

Coefficient	Value
t_{101}	$2.89\Delta\omega_3$
t_{110}	$2.89\Delta\omega_2$
t_{111}	$2.89\Delta\omega_1$
s_{101}	$-2.89\Delta W/\langle r\rangle$
s_{110}	$-2.89\Delta V/\langle r\rangle$
s_{111}	$-2.89\Delta U/\langle r\rangle$
s_{201}	$-0.65\,(\Delta M_{11}^{*} - 2\Delta M_{33}^{*})$
s_{210}	$2.24\Delta M_{23}^{+}$
s_{211}	$2.24\Delta M_{13}^{+}$
s_{220}	$2.24\Delta M_{12}^{+}$
s_{221}	$1.12\Delta M_{11}^{*}$

A kinematic analysis of the stellar velocity field is performed in the Galactic coordinate system. Since the Galactic coordinate system is obtained by transforming the stellar equatorial coordinates and proper motions of a specific catalogue, the systematic difference between the Galactic stellar proper motions of two catalogues are determined by the systematic differences between the equatorial proper motions of these catalogues. Since in our paper the systematic differences between the equatorial stellar proper motions of the TGAS and Tycho-2 catalogues are modeled by Eqs. (8), we will obtain the systematic differences between the Galactic proper motions by applying the standard procedure of determining the Galactic proper motions (ESA 1997) to them. Guided by the form of the Ogorodnikov–Milne equations (33) and (34), we decomposed the systematic Galactic stellar proper motion differences TGAS–Tycho-2 into a system of vector spherical harmonics for each of our three samples of stars. These data were used to calculate the shifts of the Ogorodnikov–Milne model parameters when passing from the Tycho-2 frame to the TGAS one. The results of these calculations are given in Table 10. Analysis of this table shows that when passing from Tycho-2 to TGAS:

- the shifts of the kinematic parameters (except for M_{13}^{+} and M_{23}^{+}) increase in absolute value with decreasing stellar brightness;
- the absolute shifts lie within the range from 0 to 4.2 km s^{-1} kpc^{-1};
- given that the rms errors of these parameters from the Tycho-2 data are 0.2 km s^{-1} kpc^{-1} (Bobylev et al. 2009), it can be said that when determining the Oort parameters, the systematics of the Tycho-2 catalogue can give shifts in the parameters exceeding their random errors.

Let us now examine the changes in Galactic rotation parameters to which the data in Table 10 will lead. Denoting the the Galactocentric distance of the Sun by R_S, for the linear velocity of the Sun relative to the Galactic center we have

$$V_S = R_S \times (A - B). \qquad (37)$$

With this velocity the Galactic rotation period in the solar neighborhood will be

$$P = \frac{2\pi R_S}{V_S}. \qquad (38)$$

In turn, the slope of the Galactic rotation curve S is determined from the formula

$$S = -(A + B). \qquad (39)$$

Table 10. Shifts of the Ogorodnikov–Milne model parameters when passing from the Tycho-2 frame to the TGAS one

Parameter shift	$G_1 = 10{-}11$	$G_2 = 11{-}12$	$G_4 = 12{-}14$
$\Delta\langle U/r\rangle$	0.45 ± 0.05	0.55 ± 0.07	0.90 ± 0.12
$\Delta\langle V/r\rangle$	-1.30 ± 0.05	-2.11 ± 0.07	-2.42 ± 0.12
$\Delta\langle W/r\rangle$	-1.05 ± 0.05	-1.81 ± 0.07	-2.29 ± 0.12
$\Delta\Omega_1$	0.62 ± 0.05	2.82 ± 0.07	4.21 ± 0.12
$\Delta\Omega_2$	1.57 ± 0.05	2.72 ± 0.07	3.38 ± 0.12
ΔB	0.04 ± 0.05	0.94 ± 0.07	1.74 ± 0.12
ΔM_{13}	0.97 ± 0.05	1.06 ± 0.07	1.00 ± 0.12
ΔM_{23}	-1.00 ± 0.05	-0.45 ± 0.07	0.25 ± 0.12
ΔA	0.71 ± 0.05	2.06 ± 0.07	3.33 ± 0.12
ΔC	0.15 ± 0.03	1.41 ± 0.04	2.01 ± 0.06
ΔK	-0.17 ± 0.06	-0.32 ± 0.08	-0.73 ± 0.13

The units are km s^{-1} kpc^{-1}.

In accordance with the epicyclic theory (Binney and Tremaine 2008; Mignard and Froeschle 2000), the Oort parameters allow the ratio of the epicyclic frequency to the angular velocity of Galactic rotation in the solar neighborhood to be estimated:

$$F = 2\sqrt{\frac{-B}{A-B}}. \tag{40}$$

Determining the mass of the Galaxy within the solar orbit (Ogorodnikov 1965) is also of great interest:

$$M = \frac{R_S \times V_S^2}{G}, \tag{41}$$

where G is the gravitational constant.

Tables 11 and 12 give the absolute and relative changes in the fundamental Oort constants A, B, C, K and the physical parameters V, P, F, S, M when passing from the Tycho-2 frame to the TGAS one for the adopted $R_S = 8$ kpc. The original results in the Tycho-2 frame were taken from Bobylev et al. (2009); the values in the TGAS frame were obtained from our estimates. Analysis of these tables shows that the absolute shifts in parameters depend on the magnitude of stars. Among the Oort parameters, C is the parameter most sensitive to the transition to the TGAS frame. Its relative error in the Tycho-2 frame reaches 100% for faint stars. The results in Table 12 show a very interesting feature. Here, we see that whereas the parameters V, P, F, M have shifts of 2–10%, the estimates of the slope of the Galactic rotation curve change from 16 to 56%.

5. CONCLUSIONS

Below we list the main results obtained in our paper.

(1) The systematic differences between the stellar positions and proper motions of the TGAS and Tycho-2 catalogues were represented as decompositions into vector spherical harmonics with allowance made for the magnitude equation. The systematic components were extracted from the individual differences with a probability of 0.977–0.999. The model of systematic differences (8), whose coefficients are given in Tables 13–18, allows any position measurements performed using Tycho-2 as a reference catalogue to be transformed to the TGAS frame.

(2) Our study of the derived systematic differences showed that they depend on the magnitude of stars. The systematic position differences can change when passing from bright, $G = 10{-}11$, to faint, $G = 12{-}13$, stars within the range from approximately -40 to 15 mas, while the systematic proper motion differences change from -3 to 3 mas yr^{-1}. The reason why the magnitude equation exists in the investigated systematic differences is that old photographic catalogues were used to determine the Tycho-2 stellar proper motions.

(3) The derived systematic stellar position and proper motion differences were studied in terms of the model of rigid-body mutual rotation and displacement of the Tycho-2 and TGAS reference frames. We showed that the basic equations of this model could be written as a linear combination of vector spherical harmonics. This immediately allows the model parameters to be directly calculated via the coefficients of the decomposition of systematic differences into vector spherical harmonics. We found that in contrast to the mutual orientation of the Hipparcos and TGAS reference frames, the orientation and mutual rotation parameters of the Tycho-2 and TGAS frames depend on the magnitude of stars. We showed that when passing from bright to faint stars, the Tycho-2 frame is turned relative to the TGAS one through an angle from 3.51 to 5.63 mas and rotates with an angular velocity from 0.35 to 1.22 mas yr^{-1}. The fact that the orientation, rotation, and displacement parameters of the Tycho-2 reference frame relative to TGAS, first, depend on the magnitude of stars and, second, differ from the rotation parameters of

Table 11. Oort constants A, B, C, K in the TGAS frame and their relative changes $R = \frac{\text{TGAS} - \text{Tycho-2}}{\text{TGAS}}$ when passing from the Tycho-2 frame to the TGAS one

		A	B	C	K
Tycho-2		15.9 ± 0.2	-12.0 ± 0.2	-3.9 ± 0.2	-2.6 ± 0.5
TGAS	$G = 10$–11	16.6 ± 0.2	-12.0 ± 0.2	-3.8 ± 0.2	-2.8 ± 0.5
R		4.3%	0%	−3.0%	6.1%
TGAS	$G = 11$–12	18.0 ± 0.2	-11.1 ± 0.2	-2.5 ± 0.2	-2.9 ± 0.5
R		11.5%	−8.5%	−56.6%	11.0%
TGAS	$G = 12$–14	19.2 ± 0.2	-10.3 ± 0.2	-1.89 ± 0.2	-3.3 ± 0.5
R		17.3%	−17.0%	−106.3%	21.9%

The first row gives the Oort constants in the Tycho-2 frame from the results of Bobylev et al. (2009). The units are km s^{-1} kpc^{-1}.

Table 12. Physical parameters V, P, S, F, M and their relative changes $R = \frac{\text{TGAS} - \text{Tycho-2}}{\text{TGAS}}$ when passing from the Tycho-2 frame to the TGAS one

Parameters		V	P	S	F	M
Tycho-2		223.2 ± 2.3	220.2 ± 2.3	3.9 ± 0.3	1.312 ± 0.008	$(9.2 \pm 0.2) \times 10^{10}$
TGAS	G_1	228.6 ± 2.4	215.0 ± 2.2	4.7 ± 0.3	1.294 ± 0.008	$(9.7 \pm 0.2) \times 10^{10}$
R		2.3%	−2.4%	16.1%	−1.4%	4.6%
TGAS	G_2	232.2 ± 2.4	211.7 ± 2.5	6.9 ± 0.3	1.235 ± 0.0098	$(10.0 \pm 0.2) \times 10^{10}$
R		3.9%	−4.0%	43.5%	−6.2%	7.6%
TGAS	G_3	235.9 ± 2.6	208.3 ± 2.6	9.0 ± 0.3	1.180 ± 0.010	$(10.3 \pm 0.2) \times 10^{10}$
R		5.4%	−5.7%	56.5%	−11.2%	10.5%

The first row gives the parameters from the results of Bobylev et al. (2009). The parameters: V is the linear rotation velocity of the Sun relative to the Galactic center (km s^{-1}); P is the revolution period of the Sun around the Galactic center (Myr); S is the slope of the Galactic rotation curve in the solar neighborhood (km s^{-1} kpc^{-1}); F is the ratio of the epicyclic frequency to the angular velocity of Galactic rotation in the solar neighborhood; M is the mass of the Galaxy within the solar orbit (in solar masses).

the Hipparcos catalogue relative to TGAS found by Lindegren et al. (2016) deserves attention. This is well explained by the fact that the Hipparcos and TGAS stellar proper motions were obtained without using ground-based catalogues.

(4) We developed a method of allowance for the influence of systematic errors in the stellar proper motions of a catalogue in the equatorial coordinate system on the determination of kinematic Ogorodnikov–Milne model parameters from the proper motions in the Galactic coordinate system. We showed that when passing from the Tycho-2 frame to the TGAS one, the largest changes are observed for the estimates of the Oort parameter C (100%) and the slope of the Galactic rotation curve in the solar neighborhood S (−56%). The corresponding changes in the Oort parameters A and B as well as the linear velocity of the Sun relative to the Galactic center, the Galactic rotation period, the ratio of the epicyclic frequency to the angular velocity of Galactic rotation, and the mass

Table 13. Spheroidal decomposition coefficients s_{nkpr} (mas yr^{-1}) of the field of position differences $\Delta\alpha\cos\delta\mathbf{e}_\alpha + \Delta\delta\mathbf{e}_\delta$ TGAS−Tycho-2 corrected for the magnitude equation; u is the ratio of the absolute value of the coefficient to its rms error

n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u
1	0	1	0	23.66	20.07	9	0	1	2	−1.24	2.73	1	1	0	1	−2.08	2.77
1	0	1	1	4.37	5.81	10	0	1	0	2.07	1.76	1	1	0	2	−0.50	1.10
1	0	1	2	−2.03	4.48	10	0	1	1	0.00	0.00	2	1	1	0	4.13	3.50
2	0	1	0	−10.76	9.12	10	0	1	2	−0.93	2.05	2	1	1	1	0.85	1.12
2	0	1	1	5.86	7.80	11	0	1	0	−2.77	2.35	2	1	1	2	−2.50	5.52
2	0	1	2	0.27	0.60	11	0	1	1	0.00	0.00	3	1	0	0	4.02	3.41
3	0	1	0	−5.49	4.65	11	0	1	2	1.24	2.74	3	1	0	1	−0.97	1.29
3	0	1	1	2.85	3.79	12	0	1	0	2.97	2.52	3	1	0	2	−1.05	2.31
3	0	1	2	0.25	0.55	12	0	1	1	0.00	0.00	4	1	0	0	2.65	2.25
4	0	1	0	−3.24	2.75	12	0	1	2	−1.33	2.93	4	1	0	2	−1.19	2.62
4	0	1	1	2.01	2.67	13	0	1	0	−0.33	0.28	5	1	1	0	2.27	1.92
4	0	1	2	−0.11	0.23	13	0	1	1	1.14	1.52	5	1	1	1	0.00	0.00
5	0	1	0	−0.28	0.24	13	0	1	2	−0.74	1.63	5	1	1	2	−1.01	2.24
5	0	1	1	0.98	1.30	14	0	1	0	−4.37	3.70	6	1	1	0	2.90	2.46
5	0	1	2	−0.63	1.40	14	0	1	1	1.14	1.52	6	1	1	1	−0.78	1.03
7	0	1	0	0.50	0.42	14	0	1	2	1.07	2.36	6	1	1	2	−0.69	1.53
7	0	1	1	−1.74	2.31	21	0	1	0	2.30	1.95	6	1	0	0	−0.23	0.20
7	0	1	2	1.12	2.47	21	0	1	1	−0.83	1.10	6	1	0	1	0.81	1.08
8	0	1	0	6.08	5.15	21	0	1	2	−0.39	0.86	6	1	0	2	−0.52	1.16
8	0	1	1	−2.27	3.02	22	0	1	0	0.38	0.32	8	1	1	0	−0.27	0.23
8	0	1	2	−0.96	2.12	22	0	1	1	−1.33	1.77	8	1	1	1	0.92	1.23
9	0	1	0	4.42	3.75	22	0	1	2	0.86	1.89	8	1	1	2	−0.60	1.32
9	0	1	1	−0.96	1.27	1	1	0	0	4.71	4.00	2	2	1	0	−0.88	0.74

Table 14. Spheroidal decomposition coefficients s_{nkpr} (mas) of the field of position differences $\Delta\alpha\cos\delta\mathbf{e}_\alpha + \Delta\delta\mathbf{e}_\delta$ TGAS−Tycho-2 corrected for the magnitude equation; u is the ratio of the absolute value of the coefficient to its rms error (the continuation of Table 13)

n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u
2	2	1	1	3.04	4.05	7	4	1	2	−0.53	1.17	9	8	1	0	2.65	2.25
2	2	1	2	−1.96	4.34	5	5	1	0	2.17	1.84	9	8	1	1	0.00	0.00
3	2	1	0	8.43	7.15	5	5	1	1	0.00	0.00	9	8	1	2	−1.19	2.62
3	2	1	1	0.76	1.01	5	5	1	2	−0.97	2.14	10	9	0	0	−2.00	1.69
3	2	1	2	−0.74	1.65	5	5	0	0	3.23	2.74	10	9	0	1	−2.59	3.45
6	2	0	0	2.51	2.13	5	5	0	1	−1.03	1.37	10	9	0	2	−1.12	2.46
6	2	0	1	0.00	0.00	5	5	0	2	−0.65	1.43	13	12	1	0	−2.09	1.78
6	2	0	2	−1.12	2.48	9	5	0	0	−2.63	2.23	13	12	1	1	−2.72	3.62
7	2	1	0	3.35	2.84	9	5	0	1	0.77	1.03	13	12	1	2	−1.17	2.58
7	2	1	1	−1.31	1.74	9	5	0	2	0.58	1.28	14	14	1	0	2.00	1.69
7	2	1	2	−0.49	1.08	10	5	0	0	2.33	1.97	14	14	1	1	2.59	3.45
3	3	0	0	2.53	2.15	10	5	0	1	0.00	0.00	14	14	1	2	1.12	2.46
3	3	0	1	0.00	0.00	10	5	0	2	−1.04	2.30	17	17	1	0	−2.27	1.93
3	3	0	2	−1.13	2.50	6	6	0	0	2.42	2.05	17	17	1	1	−2.95	3.93
5	3	1	0	2.57	2.18	6	6	0	1	0.83	1.10	17	17	1	2	−1.27	2.81
5	3	1	1	−1.18	1.58	6	6	0	2	−1.73	3.81	41	29	0	0	2.23	1.89
5	3	1	2	−0.23	0.51	10	6	0	0	2.25	1.91	41	29	0	1	0.00	0.00
4	4	1	0	−2.82	2.40	10	6	0	1	0.00	0.00	41	29	0	2	−1.00	2.20
4	4	1	1	1.00	1.34	10	6	0	2	−1.01	2.22	38	37	1	0	1.91	1.62
4	4	1	2	0.48	1.07	8	8	1	0	2.38	2.02	38	37	1	1	2.48	3.30
7	4	1	0	2.61	2.22	8	8	1	1	0.00	0.00	38	37	1	2	1.07	2.36
7	4	1	1	−0.82	1.10	8	8	1	2	−1.06	2.35					.	.

of the Galaxy within the Galactocentric distance of the Sun are not so large, being 2−10%.

Gaia Data Release 1 allowed the Tycho-2 catalogue to be supplied with stellar parallaxes and new proper motions. Undoubtedly, the addition of parallaxes for ∼2 million stars was a radical improvement of the Tycho-2 catalogue. As regards the stellar proper motions, we showed here that from the viewpoint of studying the kinematic parameters of the stellar velocity field, the new stellar proper motions obtained from the observations in space onboard the Hipparcos and Gaia spacecraft lead to a significant revision of the numerical values of the Oort parameter C and the slope of the Galactic rotation curve in the solar neighborhood. The improvements of other kinematic parameters are not so significant.

Table 15. Toroidal decomposition coefficients t_{nkpr} (mas) of the field of position differences $\Delta\alpha\cos\delta\mathbf{e}_\alpha + \Delta\delta\mathbf{e}_\delta$ TGAS−Tycho-2 corrected for the magnitude equation; u is the ratio of the absolute value of the coefficient to its rms error

n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u
1	0	1	0	9.74	8.26	2	1	0	0	3.45	2.93	9	9	1	0	−3.08	2.61
1	0	1	1	1.09	1.46	2	1	0	1	−0.77	1.03	9	9	1	1	−4.00	5.32
1	0	1	2	−0.31	0.68	2	1	0	2	−0.95	2.09	9	9	1	2	−1.72	3.80
4	0	1	0	4.66	3.95	3	1	0	0	0.35	0.30	24	23	0	0	1.95	1.65
4	0	1	1	−1.49	1.98	3	1	0	1	−1.22	1.63	24	23	0	1	2.53	3.36
4	0	1	2	−0.93	2.06	3	1	0	2	0.79	1.75	24	23	0	2	1.09	2.40
5	0	1	0	4.09	3.47	5	1	1	0	−0.27	0.22	24	23	1	0	1.69	1.43
5	0	1	1	0.00	0.00	5	1	1	1	0.92	1.22	24	23	1	1	2.20	2.92
5	0	1	2	−1.83	4.04	5	1	1	2	−0.59	1.31	24	23	1	2	0.95	2.09
6	0	1	0	−5.99	5.08	12	1	1	0	−1.67	1.41	28	28	0	0	−2.37	2.01
6	0	1	1	1.96	2.61	12	1	1	1	−2.16	2.88	28	28	0	1	0.00	0.00
6	0	1	2	1.16	2.55	12	1	1	2	−0.93	2.06	28	28	0	2	1.06	2.34
9	0	1	0	2.60	2.21	3	2	1	0	5.18	4.39	30	28	1	0	−2.09	1.78
9	0	1	1	0.00	0.00	3	2	1	1	0.00	0.00	30	28	1	1	−2.72	3.62
9	0	1	2	−1.16	2.57	3	2	1	2	−2.31	5.11	30	28	1	2	−1.17	2.58
10	0	1	0	0.23	0.19	4	2	0	0	−0.29	0.24	34	34	1	0	2.76	2.34
10	0	1	1	−0.79	1.05	4	2	0	1	0.99	1.32	34	34	1	1	3.58	4.76
10	0	1	2	0.51	1.12	4	2	0	2	−0.64	1.41	34	34	1	2	1.54	3.40
15	0	1	0	−2.69	2.29	5	2	1	0	−0.46	0.39	39	38	1	0	−1.81	1.54
15	0	1	1	0.86	1.15	5	2	1	1	1.59	2.12	39	38	1	1	−2.36	3.13
15	0	1	2	0.54	1.19	5	2	1	2	−1.03	2.27	39	38	1	2	−1.01	2.24
23	0	1	0	1.99	1.69	3	3	1	0	2.11	1.79	53	38	0	0	0.23	0.20
23	0	1	1	0.00	0.00	3	3	1	1	2.74	3.64	53	38	0	1	−0.81	1.08
23	0	1	2	−0.89	1.97	3	3	1	2	1.18	2.60	53	38	0	2	0.52	1.15
1	1	0	0	13.43	11.39	5	4	1	0	2.32	1.97	40	39	1	0	−2.67	2.27
1	1	0	1	9.14	12.16	5	4	1	1	0.00	0.00	40	39	1	1	0.00	0.00
1	1	0	2	−4.03	8.90	5	4	1	2	−1.04	2.29	40	39	1	2	1.19	2.64
1	1	1	0	4.84	4.10	7	6	1	0	−0.29	0.24	55	39	1	0	−2.23	1.89
1	1	1	1	−1.42	1.89	7	6	1	1	1.00	1.33	55	39	1	1	0.00	0.00
1	1	1	2	−1.07	2.36	7	6	1	2	−0.65	1.43	55	39	1	2	1.00	2.20

Table 16. Spheroidal decomposition coefficients s_{nkpr} (mas yr^{-1}) of the field of proper motion differences $\Delta\mu_\alpha \cos\delta \mathbf{e}_\alpha + \Delta\mu_\delta \mathbf{e}_\delta$ TGAS−Tycho-2 corrected for the magnitude equation; u is the ratio of the absolute value of the coefficient to its rms error

n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u
1	0	1	0	2.37	33.83	10	0	1	2	−0.06	1.59	2	1	1	1	−0.38	7.16
1	0	1	1	0.42	8.01	13	0	1	0	−0.32	4.53	2	1	1	2	0.06	1.68
1	0	1	2	−0.09	2.37	13	0	1	1	0.08	1.46	3	1	1	0	−0.02	0.30
2	0	1	0	−1.64	23.36	13	0	1	2	0.08	2.27	3	1	1	1	0.07	1.39
2	0	1	1	−0.29	5.47	14	0	1	0	−0.62	8.86	3	1	1	2	−0.05	1.32
2	0	1	2	0.06	1.78	14	0	1	1	−0.12	2.31	2	2	1	0	−0.30	4.27
3	0	1	0	−1.15	16.48	14	0	1	2	0.03	0.85	2	2	1	1	−0.39	7.32
3	0	1	1	−0.36	6.78	20	0	1	0	−0.28	4.00	2	2	1	2	−0.17	4.64
3	0	1	2	0.06	1.68	20	0	1	1	0.09	1.62	6	2	0	0	0.13	1.86
7	0	1	0	0.47	6.70	20	0	1	2	0.06	1.63	6	2	0	1	−0.08	1.46
7	0	1	1	−0.20	3.70	22	0	1	0	0.60	8.50	6	2	0	2	0.00	0.05
7	0	1	2	−0.06	1.61	22	0	1	1	0.00	0.08	7	2	1	0	0.18	2.56
8	0	1	0	0.28	3.97	22	0	1	2	−0.03	0.81	7	2	1	1	−0.08	1.46
8	0	1	1	−0.08	1.46	1	1	0	0	−0.68	9.78	7	2	1	2	−0.02	0.56
8	0	1	2	−0.06	1.78	1	1	0	1	−0.10	1.85	4	3	1	0	−0.18	2.60
9	0	1	0	0.78	11.08	1	1	0	2	0.07	2.00	4	3	1	1	−0.24	4.47
9	0	1	1	0.18	3.47	1	1	1	0	−0.14	2.07	4	3	1	2	−0.10	2.83
9	0	1	2	−0.05	1.46	1	1	1	1	−0.19	3.54	9	3	0	0	0.21	2.99
10	0	1	0	0.60	8.59	1	1	1	2	−0.08	2.24	9	3	0	1	−0.08	1.46
10	0	1	1	0.10	1.93	2	1	1	0	−0.45	6.48	9	3	0	2	−0.03	0.93

APPENDIX

In this paper we use real scalar spherical harmonics of the following form:

$$K_{nkp}(\alpha,\delta) = R_{nk}\begin{cases} P_{n,0}(\delta), & k=0, \quad p=1\\ P_{nk}(\delta)\sin k\alpha, & k\neq 0, \quad p=0\\ P_{nk}(\delta)\cos k\alpha, & k\neq 0, \quad p=1, \end{cases} \tag{42}$$

$$R_{nk} = \sqrt{\frac{2n+1}{4\pi}}\begin{cases}\sqrt{\frac{2(n-k)!}{(n+k)!}}, & k>0\\ 1, & k=0, \end{cases} \tag{43}$$

where α and δ are the right ascension (longitude) and declination (latitude) of a point on the celestial sphere, respectively ($0 \leq \alpha \leq 2\pi$; $-\pi/2 \leq \delta \leq \pi/2$); $P_{nk}(\delta)$ are Legendre polynomials (at $k=0$) and associated Legendre polynomials (at $k>0$), which can be calculated using the following recurrence relation:

$$P_{nk}(\delta) = \sin\delta\frac{2n-1}{n-k}P_{n-1,k}(\delta) - \frac{n+k-1}{n-k}P_{n-2,k}(\delta), \tag{44}$$

$$k=0,1,\ldots; \quad n=k+2,k+3,\ldots;$$

$$P_{kk}(\delta) = \frac{(2k)!}{2^k k!}\cos^k\delta,$$

$$P_{k+1,k}(\delta) = \frac{(2k+2)!}{2^{k+1}(k+1)!}\cos^k\delta\sin\delta.$$

When working with spherical harmonics, one index j

Table 17. Spheroidal decomposition coefficients s_{nkpr} (mas yr^{-1}) of the field of proper motion differences $\Delta\mu_\alpha \cos\delta \mathbf{e}_\alpha + \Delta\mu_\delta \mathbf{e}_\delta$ TGAS−Tycho-2 corrected for the magnitude equation; u is the ratio of the absolute value of the coefficient to its rms error (the continuation of Table 16)

n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u
4	4	0	0	−0.18	2.54	7	7	0	0	−0.23	3.23	31	30	0	0	0.02	0.30
4	4	0	1	−0.33	6.16	7	7	0	1	−0.20	3.85	31	30	0	1	−0.07	1.39
4	4	0	2	−0.07	1.83	7	7	0	2	−0.16	4.39	31	30	0	2	0.05	1.32
4	4	1	0	−0.14	2.07	8	8	1	0	−0.33	4.67	34	33	1	0	−0.17	2.42
4	4	1	1	−0.19	3.54	8	8	1	1	−0.42	8.01	34	33	1	1	−0.22	4.16
4	4	1	2	−0.08	2.24	8	8	1	2	−0.18	5.08	34	33	1	2	−0.09	2.64
5	4	1	0	−0.30	4.25	9	9	1	0	−0.21	2.96	48	36	1	0	0.02	0.30
5	4	1	1	0.08	1.46	9	9	1	1	−0.27	5.08	48	36	1	1	−0.07	1.39
5	4	1	2	0.07	2.03	9	9	1	2	−0.12	3.22	48	36	1	2	0.05	1.32
9	4	0	0	0.41	5.81	10	9	0	0	−0.15	2.11	38	37	1	0	0.25	3.59
9	4	0	1	0.11	2.08	10	9	0	1	−0.19	3.62	38	37	1	1	0.24	4.47
9	4	0	2	0.03	0.83	10	9	0	2	−0.08	2.29	38	37	1	2	0.17	4.78
6	6	0	0	−0.19	2.69	22	22	1	0	−0.14	2.07	40	39	0	0	−0.18	2.51
6	6	0	1	−0.24	4.62	22	22	1	1	−0.19	3.54	40	39	0	1	−0.23	4.31
6	6	0	2	−0.11	2.93	22	22	1	2	−0.08	2.24	40	39	0	2	−0.10	2.73
6	6	1	0	−0.29	4.18	43	29	0	0	−0.02	0.30	45	39	1	0	0.21	3.02
6	6	1	1	−0.38	7.16	43	29	0	1	0.07	1.39	45	39	1	1	−0.09	1.62
6	6	1	2	−0.16	4.54	43	29	0	2	−0.05	1.32	45	39	1	2	−0.03	0.78

is often used for their numbering, with

$$j = n^2 + 2k + p - 1. \tag{45}$$

The introduced functions satisfy the following relations:

$$\iint\limits_{\Omega} (K_i \cdot K_j)\, d\omega = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \tag{46}$$

In other words, the set of functions K_{nkp} forms an orthonormal system of functions on a sphere.

The systematic position and proper motion differences are the components of some vector field. Therefore, it seems appropriate to apply the method of decomposing this field into a system of vector spherical harmonics to study the systematic differences. In this paper we use the apparatus of vector spherical harmonics in the form in which it was applied in our previous papers on a kinematic analysis of stellar proper motions (Vityazev and Tsvetkov 2013, 2014). In this notation the toroidal, $\mathbf{T}_{nkp}$, and spheroidal, $\mathbf{S}_{nkp}$, vector spherical harmonics are specified as

$$\mathbf{T}_{nkp}(\alpha,\delta) = \frac{1}{\sqrt{n(n+1)}} \tag{47}$$
$$\times \left(\frac{\partial K_{nkp}(\alpha,\delta)}{\partial \delta} \mathbf{e}_\alpha - \frac{1}{\cos\delta} \frac{\partial K_{nkp'}(\alpha,\delta)}{\partial \alpha} \mathbf{e}_\delta \right),$$

$$\mathbf{S}_{nkp}(\alpha,\delta) = \frac{1}{\sqrt{n(n+1)}} \tag{48}$$
$$\times \left(\frac{1}{\cos\delta} \frac{\partial K_{nkp}(\alpha,\delta)}{\partial \alpha} \mathbf{e}_\alpha + \frac{\partial K_{nkp}(\alpha,\delta)}{\partial \delta} \mathbf{e}_\delta \right).$$

Denote the components of the unit vector $\mathbf{e}_\alpha$ by T^α_{nkp} and S^α_{nkp} and of the unit vector $\mathbf{e}_\delta$ by T^δ_{nkp} and S^b_{nkp}, respectively:

$$\mathbf{T}_{nkp} = T^\alpha_{nkp}\mathbf{e}_\alpha + T^\delta_{nkp}\mathbf{e}_\delta, \tag{49}$$

$$\mathbf{S}_{nkp} = S^\alpha_{nkp}\mathbf{e}_\alpha + S^\delta_{nkp}\mathbf{e}_\delta. \tag{50}$$

Given that $P_{n,k+1}(b) = 0$ at $n < k+1$, these compo-

Table 18. Toroidal decomposition coefficients t_{nkpr} (mas yr^{-1}) of the field of proper motion differences $\Delta\mu_\alpha \cos\delta \mathbf{e}_\alpha + \Delta\mu_\delta \mathbf{e}_\delta$ TGAS−Tycho-2 corrected for the magnitude equation; u is the ratio of the absolute value of the coefficient to its rms error

n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u	n	k	p	r	s_{nkpr}	u
1	0	1	0	0.99	14.13	15	0	1	1	0.07	1.23	7	1	1	2	0.03	0.93
1	0	1	1	0.13	2.39	15	0	1	2	−0.04	1.17	9	1	0	0	−0.13	1.89
1	0	1	2	0.00	0.07	18	0	1	0	0.16	2.23	9	1	0	1	−0.17	3.24
2	0	1	0	−0.18	2.52	18	0	1	1	0.10	1.93	9	1	0	2	−0.07	2.05
2	0	1	1	0.13	2.54	18	0	1	2	0.12	3.42	2	2	1	0	0.48	6.90
2	0	1	2	−0.03	0.71	1	1	1	0	0.28	3.95	2	2	1	1	0.07	1.39
3	0	1	0	−0.21	3.05	1	1	1	1	−0.18	3.31	2	2	1	2	0.01	0.15
3	0	1	1	−0.28	5.24	1	1	1	2	0.01	0.34	3	2	1	0	−0.71	10.20
3	0	1	2	−0.12	3.32	1	1	0	0	−3.46	49.43	3	2	1	1	−0.49	9.17
4	0	1	0	1.05	15.04	1	1	0	1	−1.09	20.64	3	2	1	2	0.06	1.76
4	0	1	1	0.10	1.85	1	1	0	2	0.23	6.44	7	2	0	0	0.12	1.66
4	0	1	2	−0.03	0.78	2	1	1	0	−0.12	1.71	7	2	0	1	0.15	2.85
5	0	1	0	−0.36	5.16	2	1	1	1	−0.16	2.93	7	2	0	2	0.07	1.81
5	0	1	1	−0.47	8.86	2	1	1	2	−0.07	1.85	9	2	1	0	0.12	1.72
5	0	1	2	−0.20	5.61	3	1	0	0	0.20	2.87	9	2	1	1	−0.08	1.46
6	0	1	0	−0.92	13.12	3	1	0	1	0.26	4.93	9	2	1	2	0.01	0.17
6	0	1	1	−0.08	1.54	3	1	0	2	0.11	3.12	5	4	1	0	−0.24	3.47
6	0	1	2	0.04	1.22	4	1	0	0	0.44	6.22	5	4	1	1	−0.25	4.70
10	0	1	0	0.72	10.31	4	1	0	1	0.06	1.16	5	4	1	2	0.04	1.17
10	0	1	1	0.11	2.08	4	1	0	2	0.00	0.12	9	5	0	0	−0.02	0.27
10	0	1	2	−0.01	0.27	5	1	1	0	−0.19	2.69	9	5	0	1	0.07	1.23
13	0	1	0	0.02	0.27	5	1	1	1	−0.24	4.62	9	5	0	2	−0.04	1.17
13	0	1	1	−0.07	1.23	5	1	1	2	−0.11	2.93	7	7	1	0	0.19	2.74
13	0	1	2	0.04	1.17	7	1	1	0	−0.21	2.99	7	7	1	1	0.25	4.70
15	0	1	0	−0.02	0.27	7	1	1	1	0.08	1.46	7	7	1	2	0.11	2.98

nents as defined as

$$T^{\alpha}_{nkp} = S^{\delta}_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \tag{51}$$

$$\times \begin{cases} P_{n,1}(\delta), \\ k=0, \quad p=1 \\ (-k\tan\delta P_{nk}(\delta) + P_{n,k+1}(\delta))\sin k\alpha, \\ k\neq 0, \quad p=0 \\ (-k\tan\delta P_{nk}(\delta) + P_{n,k+1}(\delta))\cos k\alpha, \\ k\neq 0, \quad p=1; \end{cases}$$

$$T^{\delta}_{nkp} = -S^{\alpha}_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \tag{52}$$

$$\times \begin{cases} 0, & k=0, \quad p=1 \\ -\frac{k}{\cos\delta}P_{nk}(\delta)\cos k\alpha, & k\neq 0, \quad p=0 \\ +\frac{k}{\cos\delta}P_{nk}(\delta)\sin k\alpha, & k\neq 0, \quad p=1. \end{cases}$$

The introduced functions form an orthonormal system of functions on a sphere:

$$\iint\limits_{\Omega} (\mathbf{T}_i \cdot \mathbf{T}_j)\, d\omega \tag{53}$$

$$= \iint_\Omega (\mathbf{S}_i \cdot \mathbf{S}_j)\, d\omega = \begin{cases} 0, & i \neq j \\ 1, & i = j; \end{cases}$$

$$\iint_\Omega (\mathbf{S}_i \cdot \mathbf{T}_j)\, d\omega = 0, \quad \forall i, j. \qquad (54)$$

Normalized Legendre polynomials are used in Eq. (8) to describe the magnitude equation:

$$t_{nkp}(G) = \sum_r t_{nkpr} Q_r(\bar{G}); \qquad (55)$$

$$s_{nkp}(G) = \sum_r s_{nkpr} Q_r(\bar{G}),$$

where the transformation

$$\bar{G} = 2\frac{G - G_{\min}}{G_{\max} - G_{\min}} - 1 \qquad (56)$$

transforms the segment $[G_{\min} \leq G \leq G_{\max}]$ to the segment $[-1 \leq \bar{m} \leq +1]$.

Normalized Legendre polynomials are used in Eq. (55):

$$Q_r(\bar{G}) = \sqrt{\frac{2r+1}{2}} P_r(\bar{G}), \qquad (57)$$

while $P_r(\bar{G})$ are Legendre polynomials; the following recurrence relation can be used to calculate the latter:

$$P_{r+1}(\bar{G}) = \frac{2r+1}{r+1}\bar{G}P_r(\bar{G}) - \frac{r}{r+1}P_{r-1}(\bar{G}), \quad (58)$$

$$r = 1, 2, \ldots, \quad P_0 = 1, \quad P_1 = \bar{G}.$$

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