ISSN 1063-7737, Astronomy Letters, 2014, Vol. 40, No. 11, pp. 713–723. © Pleiades Publishing, Inc., 2014.
Original Russian Text © V.V. Vityazev, A.S. Tsvetkov, D.A. Trofimov, 2014, published in Pis'ma v Astronomicheskiĭ Zhurnal, 2014, Vol. 40, No. 11, pp. 783–793.

Kinematic Analysis of Stellar Radial Velocities by the Spherical Harmonics

V. V. Vityazev*, **A. S. Tsvetkov****, and **D. A. Trofimov*****

St. Petersburg State University, Bibliotechnaya pl. 2, St. Petersburg, 198504 Russia

Received April 21, 2014

Abstract—A method for a kinematic analysis of stellar radial velocities using spherical harmonics is proposed. This approach does not depend on the specific kinematic model and allows both low-frequency and high-frequency kinematic radial velocity components to be analyzed. The possible systematic variations of distances with coordinates on the celestial sphere that, in turn, are modeled by a linear combination of spherical harmonics are taken into account. Theoretical relations showing how the coefficients of the decomposition of distances affect the coefficients of the decomposition of the radial velocities themselves have been derived. It is shown that the larger the mean distance to the sample of stars being analyzed, the greater the shift in the solar apex coordinates, while the shifts in the Oort parameter A are determined mainly by the ratio of the second zonal harmonic coefficient to the mean distance to the stars, i.e., by the degree of flattening of the spatial distribution of stars toward the Galactic plane. The distances to the stars for which radial velocity estimates are available in the CRVAD-2 catalog have been decomposed into spherical harmonics, and the existing variations of distances with coordinates are shown to exert no noticeable influence on both the solar motion components and the estimates of the Oort parameter A, because the stars from this catalog are comparatively close to the Sun (no farther than 500 pc). In addition, a kinematic component that has no explanation in terms of the three-dimensional Ogorodnikov–Milne model is shown to be detected in the stellar radial velocities, as in the case of stellar proper motions.

DOI: 10.1134/S1063773714110073

Keywords: *astrometry, stellar radial velocities and proper motions, parallaxes, spherical harmonics, stellar kinematics, Hipparcos.*

INTRODUCTION

A kinematic analysis of the stellar velocity field requires knowing the stellar coordinates, proper motions, parallaxes (distances), and radial velocities. The data on the photometry, spectral types, and color indices of stars are also very useful. At present, there are astrometric catalogs containing information about the positions and proper motions for hundreds of millions of stars: UCAC4 (Zacharias et al. 2013), PPMXL (Roeser et al. 2010), and XPM (Fedorov et al. 2009). In contrast, the number of stars with measured trigonometric parallaxes slightly exceeds 100 000 (Hipparcos, Perryman et al. 1997; van Leeuwen 2007). The OSACA (Bobylev et al. 2006) and CRVAD-2 (Kharchenko et al. 2007) catalogs of radial velocities containing no more than 55 000 stars are even less rich. The absence of information about the trigonometric parallaxes of stars is not fatal for a kinematic analysis of stellar proper motions and radial velocities. First, instead of the trigonometric parallaxes, their analogs, the photometric parallaxes, can occasionally be used; second, even in the complete absence of data on the parallaxes, a partial solution of the problem is possible, since instead of the true solar motion components (when analyzing the stellar proper motions) and instead of the stellar velocity deformation field parameters (when analyzing the radial velocities), their values can be determined to within a constant factor equal to the mean parallax or mean distance for the sample of stars being analyzed. However, the parallaxes of the sample of stars under consideration can have systematic variations over the celestial sphere. As Olling and Dehnen (2003) showed, when the stellar proper motions are analyzed, this leads to a distortion of the sought-for kinematic model parameters due to mode mixing (according to their terminology). They studied this effect using a simplified kinematic model in which no effects in planes perpendicular to the principal Galactic plane were included. In addition, mode mixing was studied in the one-dimensional case of the dependence of parallaxes

*E-mail: vityazev@list.ru
**E-mail: astsvetkov@inbox.ru
***E-mail: dm.trofimov@gmail.com

only on longitude. In view of these simplifications, the main mathematical apparatus in the cited paper was the Fourier series to represent both the proper motions of stars and their parallaxes. This approach limited the possibility of allowance for the dependence of parallaxes on longitude to a narrow zone of latitudes near the Galactic equator. Previously (Vityazev and Tsvetkov 2013, 2014), we solved this problem when analyzing the stellar proper motions in terms of the three-dimensional Ogorodnikov—Milne model using spherical harmonics instead of Fourier series. This allowed the mode-mixing effect to be studied not only in longitude but also in latitude.

As far as we know, an analogous problem has not yet been posed when analyzing the radial velocities. For this reason, the present paper is devoted to investigating the influence of the systematic variations of distances over the celestial sphere on the determination of the coefficients of the decomposition of stellar radial velocities into spherical harmonics. This approach allows both the parameters of the standard kinematic Ogorodnikov—Milne model and the higher harmonics that do not enter into the standard model to be determined. For this purpose, we derive formulas showing how the coefficients of the decomposition of distances into spherical harmonics affect the determination of the coefficients of the decomposition of the radial velocities themselves into spherical harmonics and the Ogorodnikov—Milne model parameters. We find the conditions under which the systematic variations of distances over the celestial sphere exert a significant influence on the determination of the solar apex coordinates and the Oort parameter $A = M^+_{12}$. The theoretical results obtained are compared with the data from a kinematic analysis of the radial velocities from the CRVAD-2 catalog (Kharchenko et al. 2007), which also contains the equatorial coordinates (J2000), proper motions, and trigonometric parallaxes in the Hipparcos system, the B and V photometry in Johnson's system, and the spectral types of stars.

THE LINEAR OGORODNIKOV—MILNE MODEL

The equations of the linear Ogorodnikov-Milne model (Ogorodnikov 1965; du Mont 1977) are commonly used to analyze the stellar velocity field. In this model, the stellar velocity field is represented by the linear expression

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{\Omega} \times \mathbf{r} + \mathbf{M}^+ \mathbf{r}, \qquad (1)$$

where $\mathbf{V}$ is the stellar velocity, $\mathbf{V}_0$ is the influence of the translational motion of the Sun, $\mathbf{\Omega}$ is the angular velocity of rigid-body rotation of the stellar system, and $\mathbf{M}^+$ is the symmetric deformation tensor of the velocity field. The Ogorodnikov—Milne model contains 12 parameters:

U, V, W are the components of the translational velocity vector of the Sun $\mathbf{V}_0$ relative to the stars;

$\omega_1, \omega_2, \omega_3$ are the components of the rigid-body rotation vector $\mathbf{\Omega}$;

$M^+_{11}, M^+_{22}, M^+_{33}$ are the parameters of the tensor $\mathbf{M}^+$ describing the velocity field contraction—expansion along the principal Galactic axes;

$M^+_{12}, M^+_{13}, M^+_{23}$ are the parameters of the tensor $\mathbf{M}^+$ describing the velocity field deformation in the principal plane and two planes perpendicular to it.

Projecting Eq. (1) onto the unit vectors of the Galactic coordinate system and introducing the factor $\mathcal{K} = 4.74$ to convert the dimensions of stellar proper motions mas yr^{-1} into km s^{-1} kpc^{-1}, we obtain

$$\begin{aligned} \mathcal{K}\mu_l \cos b = {} & U/r \sin l - V/r \cos l \qquad (2)\\ & - \omega_1 \sin b \cos l - \omega_2 \sin b \sin l + \omega_3 \cos b \\ & - M^+_{13} \sin b \sin l + M^+_{23} \sin b \cos l \\ & + M^+_{12} \cos b \cos 2l - \frac{1}{2} M^*_{11} \cos b \sin 2l, \end{aligned}$$

$$\begin{aligned} \mathcal{K}\mu_b = {} & U/r \cos l \sin b + V/r \sin l \sin b \qquad (3)\\ & - W/r \cos b + \omega_1 \sin l - \omega_2 \cos l \\ & - \frac{1}{2} M^+_{12} \sin 2b \sin 2l + M^+_{13} \cos 2b \cos l \\ & + M^+_{23} \cos 2b \sin l \\ & - \frac{1}{4} M^*_{11} \sin 2b \cos 2l + \frac{1}{2} X \sin 2b, \end{aligned}$$

$$\begin{aligned} V_r = {} & -U \cos l \cos b - V \sin l \cos b \qquad (4)\\ & - W \sin b + r(M^+_{13} \sin 2b \cos l \\ & + M^+_{23} \sin 2b \sin l + M^+_{12} \cos^2 b \sin 2l \\ & + M^+_{11} \cos^2 b \cos^2 l + M^+_{22} \cos^2 b \sin^2 l + M^+_{33} \sin^2 b). \end{aligned}$$

In these formulas,

$$\begin{aligned} M^*_{11} &= M^+_{11} - M^+_{22}, \qquad (5)\\ X &= M^+_{33} - \frac{1}{2}(M^+_{11} + M^+_{22}). \end{aligned}$$

SCALAR SPHERICAL HARMONICS

Spherical harmonics are widely used in various fields of mathematics and physics; their definition can be found in many sources (see, e.g., Arfken 1966). In

Table 1. Coefficients of the decomposition of the model radial velocities (4) into spherical harmonics at $r = \hat{r}$

Coefficient	Value
v_{001}	$1.18\hat{r}(M_{11}^{+} + M_{22}^{+} + M_{33}^{+})$
v_{101}	$-2.05W$
v_{110}	$-2.05V$
v_{111}	$-2.05U$
v_{201}	$-0.53\hat{r}(M_{11}^{+} + M_{22}^{+} - 2M_{33}^{+})$
v_{210}	$1.83M_{23}^{+}\hat{r}$
v_{211}	$1.83M_{13}^{+}\hat{r}$
v_{220}	$1.83M_{12}^{+}\hat{r}$
v_{221}	$0.92\hat{r}(M_{11}^{+} - M_{22}^{+})$

this paper, we will use the following representation for spherical harmonics:

$$K_{nkp}(l,b) \tag{6}$$
$$= R_{nk}\begin{cases} P_{n,0}(b), & k=0, \quad p=1; \\ P_{nk}(b)\sin kl, & k\neq 0, \quad p=0; \\ P_{nk}(b)\cos kl, & k\neq 0, \quad p=1, \end{cases}$$

$$R_{nk} = \sqrt{\frac{2n+1}{4\pi}}\begin{cases} \sqrt{\frac{2(n-k)!}{(n+k)!}}, & k>0; \\ 1, & k=0. \end{cases} \tag{7}$$

In Eq. (6), the longitude and latitude of a point on the sphere ($0 \le l \le 2\pi$; $-\pi/2 \le b \le \pi/2$) are denoted by l and b, respectively; the Legendre polynomials (at $k = 0$) and associated Legendre functions (at $k > 0$) are denoted by $P_{nk}(b)$. They can be calculated using the following recurrence relations:

$$P_{nk}(b) = \sin b\frac{2n-1}{n-k}P_{n-1,k}(b) \tag{8}$$
$$-\frac{n+k-1}{n-k}P_{n-2,k}(b), \quad {}_{n=k+2,k+3,\ldots}^{k=0,1,\ldots}$$
$$P_{kk}(b) = \frac{(2k)!}{2^k k!}\cos^k b,$$
$$P_{k+1,k}(b) = \frac{(2k+2)!}{2^{k+1}(k+1)!}\cos^k b\sin b.$$

For convenience, a linear numbering of the functions $\mathbf{V}_{nkp}$ by one index j is often introduced:

$$j = n^2 + 2k + p - 1. \tag{9}$$

The introduced functions satisfy the relations

$$\iint\limits_{\Omega}(K_iK_j)\,d\omega = \begin{cases} 0, & i\neq j; \\ 1, & i=j. \end{cases} \tag{10}$$

In other words, the set of functions K_{nkp} forms an orthonormal system of functions on the sphere.

ANALYSIS OF STELLAR RADIAL VELOCITIES WHEN THE STELLAR DISTANCES ARE UNKNOWN

Strictly speaking, to perform a kinematic analysis of stars, their parallaxes or distances should be known. In those cases where the distances are unknown (this is more likely a rule than an exception), we have to assume in Eq. (4) that all stars are at the same distance from us, which will be denoted by $\hat{r}$. In this case, we will be able to determine not the parameters M_{pq}^{+} themselves but the quantities $\hat{r}M_{pq}^{+}$, $p,q = 1,2,3$ from Eq. (4). Using the expression

$$v_{nkp} = \iint\limits_{\Omega}(V_rK_{nkp})\,d\omega \tag{11}$$
$$= \int_0^{2\pi} dl \int_{-\pi/2}^{+\pi/2} V_r(l,b)K_{nkp}(l,b)\cos b db,$$

we calculated the coefficients of the decomposition of the right part of Eq. (4) into a system of spherical harmonics at $r = \hat{r}$. The results of our calculations are presented in Table 1.

To solve the inverse problem, i.e., to determine the Ogorodnikov–Milne model parameters via the coefficients of the decomposition of stellar radial velocities into vector spherical harmonics, we can use the formulas given in Table 2.

THE SYSTEMATIC VARIATIONS OF DISTANCES WITH COORDINATES BASED ON DATA FROM THE CRVAD-2 CATALOG

To estimate the degree of distortions introduced into the results of our kinematic analysis of stellar radial velocities by the variations of stellar distances with coordinates, we will use data from the CRVAD-2 catalog (Kharchenko et al. 2007), which contains the radial velocities and highly accurate astrometry for 54 907 stars. Only 48 784 stars can be used for this analysis, because there are no parallax estimates for 2188 stars, while the parallaxes for 3935 stars are negative. Since there are ∼3000 stars with distances >500 pc among the 48 784 stars, we restricted ourselves to a sample of 45 803 stars in the range of distances from 90 to 450 pc with the mean distance

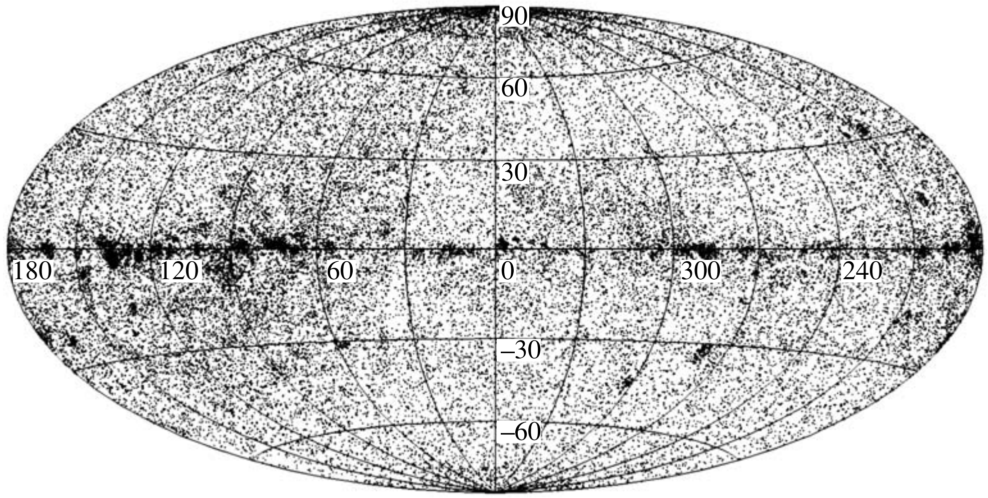

Fig. 1. Distribution of 45 803 stars from the CRVAD-2 catalog over the celestial sphere in Galactic coordinates.

$\hat{r} = 194$ pc. Figure 1 shows the distribution of these stars over the celestial sphere in Galactic coordinates. Star density fluctuations are observed in both latitude and longitude.

Obviously, nonuniformity of the distribution of stars over the celestial sphere can cause systematic variations of stellar distances with coordinates. To show this, let us represent the dependence of stellar distances on Galactic coordinates using the spherical harmonic decomposition

$$r(l, b) = \sum_{nkp} r_{nkp} K_{nkp}(l, b). \quad (12)$$

In what follows, we will assume

$$\hat{r} = \frac{1}{4\pi} \iint_{\Omega} r(l, b) d\omega \quad (13)$$
$$= K_{001} r_{001} = 0.282 r_{001}$$

to be the mean distance to the stars.

Let us calculate the coefficients r_{nkp} for our sample of stars. The results obtained are given in Table 3, where the statistically significant (according to the 3σ criterion) coefficients are highlighted in boldface. Figure 2 presents isolines of the deviations from the mean distance to the stars in Galactic coordinates; only the statistically significant harmonics were taken into account when constructing the isolines. In addition, Figs. 3 and 4 show the same data obtained from the even and odd coefficients r_{nkp}, respectively. Figures 2 and 3 demonstrate the following characteristic features of the distribution of stellar distances in coordinates. First, there is an increase in the distances as one moves from the poles in latitude. This effect is easily explained by the finite thickness of the Galactic disk, as a result of which more distant stars are seen at low latitudes than those at high latitudes despite the interstellar extinction. In decomposition (12), this circumstance is determined by the zonal part of the decomposition, i.e., by longitude-independent harmonics (K_{001}, K_{201}, K_{401}, ...). Second, a double wave in longitude is observed in the decomposition of stellar distances into even harmonics, because systematic variations of distances are clearly seen, with the stars with the largest distances are low latitudes being grouped at the longitudes $l = 90°$ and $270°$ (the positive and negative directions of the Y axis, i.e., in the direction of Galactic rotation and in the opposite

Table 2. Relations of the Ogorodnikov–Milne model parameters to the coefficients of the decomposition of stellar radial velocities into vector spherical harmonics

Parameter	Formula
W	$-0.488 v_{101}$
V	$-0.488 v_{110}$
U	$-0.488 v_{111}$
$M^+_{11}\hat{r}$	$0.282 v_{001} - 0.314 v_{201} + 0.543 v_{221}$
$M^+_{22}\hat{r}$	$0.282 v_{001} - 0.314 v_{201} - 0.543 v_{221}$
$M^+_{33}\hat{r}$	$0.282 v_{001} + 0.629 v_{201}$
$M^+_{23}\hat{r}$	$0.546 v_{210}$
$M^+_{13}\hat{r}$	$0.546 v_{211}$
$M^+_{12}\hat{r}$	$0.546 v_{220}$

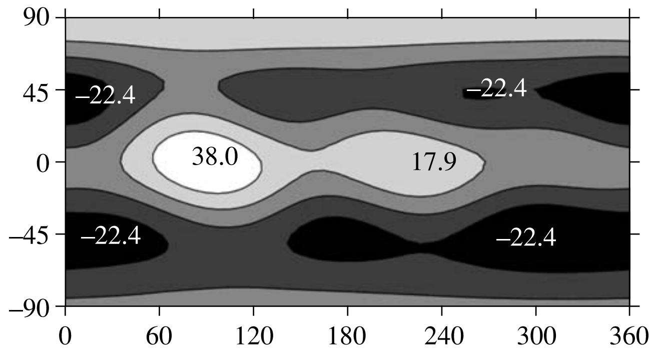

Fig. 2. Isolines of the deviations from the mean distance to 45 803 stars from the CRVAD-2 catalog over the celestial sphere in Galactic coordinates (%). The Galactic longitude and latitude in degrees are along the horizontal and vertical axes, respectively.

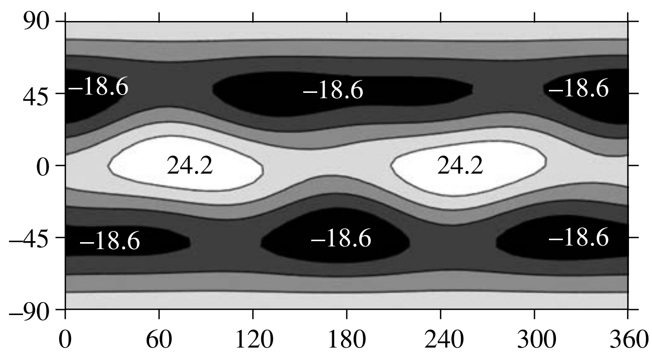

Fig. 3. Isolines of the deviations from the mean distance to 45 803 stars from the CRVAD-2 catalog over the celestial sphere in Galactic coordinates (%). The Galactic longitude and latitude in degrees are along the horizontal and vertical axes, respectively. Only the even coefficients r_{nkp} were taken into account.

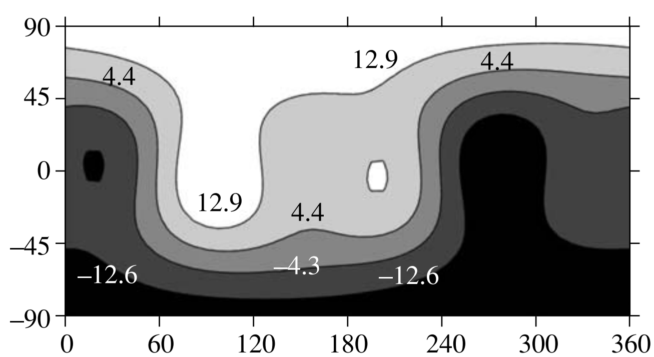

Fig. 4. Isolines of the deviations from the mean distance to 45 803 stars from the CRVAD-2 catalog over the celestial sphere in Galactic coordinates (%). The Galactic longitude and latitude in degrees are along the horizontal and vertical axes, respectively. Only the odd coefficients r_{nkp} were taken into account.

direction). On the one hand, the decomposition into odd harmonics shows an asymmetry in the distribution of distances in latitude (the influence of the harmonics K_{101} and K_{301}); on the other hand, the stars with the largest distances at low latitudes are grouped at $l = 90°$, while the stars with the smallest distances are grouped near $l = 270°$ (the influence of the harmonics K_{110}, K_{111}, and K_{330}).

ANALYSIS OF STELLAR RADIAL VELOCITIES WHEN THE SYSTEMATIC VARIATIONS OF STELLAR DISTANCES OVER THE CELESTIAL SPHERE ARE TAKEN INTO ACCOUNT

The results presented in Tables 1 and 2 can be used in those cases where the stars are within comparatively narrow spherical shells and deviate from the mean distance randomly. However, as has been shown in the preceding section, the stellar

Table 3. Coefficients of the decomposition of the distances to 45 803 stars from the CRVAD-2 catalog into spherical harmonics, Eq. (12)). The units of measurement are pc.

j	n	k	p	r_{nkp}	$\sigma_{r_{nkp}}$
0	0	0	1	**687.986**	6.417
1	1	0	1	**36.036**	6.401
2	1	1	0	**46.382**	6.425
3	1	1	1	**−40.003**	6.425
4	2	0	1	**−69.572**	6.362
5	2	1	0	8.524	6.439
6	2	1	1	−9.101	6.441
7	2	2	0	**19.494**	6.435
8	2	2	1	**−40.349**	6.408
9	3	0	1	**19.590**	6.356
10	3	1	0	10.740	6.417
11	3	1	1	−2.981	6.424
12	3	2	0	7.376	6.445
13	3	2	1	17.309	6.440
14	3	3	0	**−29.596**	6.414
15	3	3	1	3.200	6.421
16	4	0	1	**91.779**	6.292
17	4	1	0	3.980	6.456
18	4	1	1	4.301	6.464
19	4	2	0	−7.206	6.417
20	4	2	1	12.124	6.404
21	4	3	0	−5.976	6.429
22	4	3	1	**−21.231**	6.466
23	4	4	0	17.933	6.458
24	4	4	1	−17.217	6.371

distances can have systematic variations over the celestial sphere, and this should be taken into account when performing a kinematic analysis of stellar radial velocities. Let us perform a decomposition of Eq. (4) in which the distances have systematic variations with coordinates on the sphere. For this purpose, we will make calculations using Eq. (14) by taking into account the systematic variations of distances specified by decomposition (12). The results obtained are given in Tables 4 and 5. The relation $r_{001} = 3.545\hat{r}$ following from (13) was taken into account in these tables.

It can be seen from Tables 4 and 5 that theoretically all parameters of the Ogorodnikov–Milne model are plagued with the systematic variations of distances over the celestial sphere. First of all, note that the coefficients r_{nkp} odd in index n affect only the determination of the solar motion components. Accordingly, this allows the solar apex coordinates to be calculated. Let us take (10, 15, 7) km s^{-1} for the solar motion components (U, V, W). The following apex longitude and latitude correspond to these values: $L = 62\overset{\circ}{.}241$ and $B = 18\overset{\circ}{.}057$. Let us estimate the change in these coordinates due to the influence of the odd coefficients r_{110}, r_{111}, and r_{320} of the decomposition of distances into spherical harmonics. For this purpose, let us produce a model catalog of radial velocities using Eq. (4) by taking these solar motion components and $M_{12}^{+} = 15$ in it. We will assume all of the remaining parameters to be zero. The calculated radial velocities are attributed to the centers $l_t, b_t, t = 1, ..., 432$ of the spherical trapeziums obtained by dividing the Galactic equator and the latitude circle into 24 and 18 parts, respectively. Now, we determine the coefficients v_{nkp} by solving the following equations by the least-squares method:

$$V_r(l_t, b_t) = \sum_{nkp} v_{nkp} K_{nkp}(l_t, b_t), \qquad (14)$$
$$t = 1, ..., 432,$$

by assigning the weight $\cos b_t$ to each equation to compensate for the crowding of the centers of our trapeziums toward the poles. Using the formulas from Table 2, we will then determine the Ogorodnikov–Milne model parameters of interest. The results of these calculations are given in Table 6. Here, the assumption is made that the noise component is generated not so much by the random error in the radial velocities (according to the CRVAD-2 catalog, approximately 2 km s^{-1}) as by the "cosmic dispersion," i.e., by the scatter of residual velocities. By the residual velocity we mean the difference between the observed and model stellar velocities specified by Eq. (4). Our analysis of various samples of stars from the CRVAD-2 catalog showed that the

Table 4. Coefficients of the decomposition of stellar radial velocities into spherical harmonics when Eq. (12) is taken into account

Coefficients of decomposition of radial velocities	Dependence of coefficients v_{nkp} on coefficients of decomposition of distances r according to Eq. (12)
v_{001}	$+M_{13}^{+}(0.52r_{211}+\ldots)$
	$+M_{23}^{+}(0.52r_{210}+\ldots)$
	$+M_{12}^{+}(0.52r_{220}+\ldots)$
	$+M_{11}^{+}(0.33r_{001}-0.15r_{201}+0.26r_{221}+\ldots)$
	$+M_{22}^{+}(0.33r_{001}-0.15r_{201}-0.26r_{221}+\ldots)$
	$+M_{33}^{+}(0.33r_{001}+0.30r_{201}+\ldots)$
v_{101}	$+M_{13}^{+}(0.40r_{111}+0.43r_{311}+\ldots)$
	$+M_{23}^{+}(0.40r_{110}+0.43r_{310}+\ldots)$
	$+M_{12}^{+}(0.34r_{320}+\ldots)$
	$+M_{11}^{+}(0.20r_{101}-0.13r_{301}+0.17r_{321}+\ldots)$
	$+M_{22}^{+}(0.20r_{101}-0.13r_{301}-0.17r_{321}+\ldots)$
	$+M_{33}^{+}(0.60r_{101}+0.26r_{301}+\ldots)$
v_{110}	$+M_{13}^{+}(0.34r_{320}+\ldots)$
	$+M_{23}^{+}(0.40r_{101}-0.26r_{310}-0.34r_{321}+\ldots)$
	$+M_{12}^{+}(0.40r_{111}-0.11r_{311}-0.41r_{331}+\ldots)$
	$+M_{11}^{+}(0.20r_{110}-0.05r_{310}+0.21r_{330}+\ldots)$
	$+M_{22}^{+}(0.60r_{110}-0.16r_{310}-0.21r_{330}+\ldots)$
	$+M_{33}^{+}(0.20r_{110}+0.21r_{310}+\ldots)$
v_{111}	$+M_{13}^{+}(0.40r_{101}-0.26r_{301}+0.34r_{321}+\ldots)$
	$+M_{23}^{+}(0.34r_{320}+\ldots)$
	$+M_{12}^{+}(0.40r_{110}-0.11r_{310}+0.41r_{330}+\ldots)$
	$+M_{11}^{+}(0.60r_{111}-0.16r_{311}+0.21r_{331}+\ldots)$
	$+M_{22}^{+}(0.20r_{111}-0.05r_{311}-0.21r_{331}+\ldots)$
	$+M_{33}^{+}(0.20r_{111}+0.21r_{311}+\ldots)$

rms value of the residual radial velocity component is within the range $\sigma_0 = 30{-}40$ km s^{-1}. It can be shown that if, on average, there are 100 stars for each of the 432 spherical trapeziums, then the rms error of the differences ΔL and ΔB reaches 0.5°. Therefore, as can be seen from Table 6, the larger the mean distance to the sample of stars being analyzed and the wider the range of distances in which the stars are located, the greater the shift in the solar apex coordinates. This effect is small and is lost in the noise for nearby stars, which is observed for the sample of 45 803 stars from the CRVAD-2 catalog.

In contrast to the "solar" terms in Eq. (4), it can be seen from Tables 4 and 5 that the deformation tensor elements are affected only by the coefficients r_{nkp} with even n. In those cases where the distances are unknown, these components can be determined to within a factor equal to the mean distance to our sample of stars. Let us estimate the distortions introduced by the coefficient r_{201} into the Oort parameter

Table 5. Coefficients of the decomposition of stellar radial velocities into spherical harmonics when Eq. (12) is taken into account

Coefficients of decomposition of radial velocities	Dependence of coefficients v_{nkp} on coefficients of decomposition of distances r according to Eq. (12)
v_{201}	$+M_{13}^{+}(0.16r_{211}+0.40r_{411}+\ldots)$ $+M_{23}^{+}(0.16r_{210}+0.40r_{410}+\ldots)$ $+M_{12}^{+}(-0.33r_{220}+0.29r_{420}+\ldots)$ $+M_{11}^{+}(-0.15r_{001}+0.24r_{201}-0.17r_{221}$ $-0.13\cdot r_{401}+0.14r_{421}+\ldots)$ $+M_{22}^{+}(-0.15r_{001}+0.24r_{201}+0.17r_{221}$ $-0.13r_{401}-0.14r_{421}+\ldots)$ $+M_{33}^{+}(0.30r_{001}+0.52r_{201}+0.26r_{401}+\ldots)$
v_{210}	$+M_{13}^{+}(0.29r_{220}+0.33r_{420}+\ldots)$ $+M_{23}^{+}(0.52r_{001}+0.17r_{201}-0.29r_{221}$ $-0.30r_{401}-0.33r_{421}+\ldots)$ $+M_{12}^{+}(0.29r_{221}-0.12r_{411}-0.31r_{431}+\ldots)$ $+M_{11}^{+}(0.14r_{210}-0.06r_{410}+0.15r_{430}+\ldots)$ $+M_{22}^{+}(0.43r_{210}-0.18tr_{410}-0.15r_{430}+\ldots)$ $+M_{33}^{+}(0.43r_{210}+0.23r_{410}+\ldots)$
v_{211}	$+M_{13}^{+}(0.52r_{001}+0.17r_{201}$ $+0.29r_{221}-0.30r_{401}+0.33r_{421}+\ldots)$ $+M_{23}^{+}(0.29r_{220}+0.33r_{420}+\ldots)$ $+M_{12}^{+}(0.29r_{210}-0.12r_{410}+0.31r_{430}+\ldots)$ $+M_{11}^{+}(0.43r_{211}-0.18r_{411}+0.15r_{431}+\ldots)$ $+M_{22}^{+}(0.14r_{211}-0.06r_{411}-0.15r_{431}+\ldots)$ $+M_{33}^{+}(0.43r_{211}+0.23r_{411}+\ldots)$
v_{220}	$+M_{13}^{+}(0.29r_{210}-0.12r_{410}+0.31r_{430}+\ldots)$ $+M_{23}^{+}(0.29r_{211}-0.12r_{411}-0.31r_{431}+\ldots)$ $+M_{12}^{+}(0.52r_{001}-0.33r_{201}+0.07r_{401}$ $-0.44r_{441}+\ldots)$ $+M_{11}^{+}(0.43r_{220}-0.08r_{420}+0.22r_{440}+\ldots)$ $+M_{22}^{+}(0.43r_{220}-0.08r_{420}-0.22r_{440}+\ldots)$ $+M_{33}^{+}(0.14r_{220}+0.17r_{420}+\ldots)$
v_{221}	$+M_{13}^{+}(0.29r_{211}-0.12r_{411}+0.31r_{431}+\ldots)$ $+M_{23}^{+}(-0.29r_{210}+0.12r_{410}+0.31r_{430}+\ldots)$ $+M_{12}^{+}(0.44r_{440}+\ldots)$ $+M_{11}^{+}(0.26r_{001}-0.17r_{201}+0.43r_{221}$ $+0.04r_{401}-0.08r_{421}+0.22r_{441}+\ldots)$ $+M_{22}^{+}(-0.26r_{001}+0.17r_{201}+0.43r_{221}$ $-0.04r_{401}-0.08r_{421}-0.22r_{441}+\ldots)$ $+M_{33}^{+}(0.14r_{221}+0.17r_{421}+\ldots)$

Table 6. Influence of the coefficients r_{001}, r_{110}, r_{111}, and r_{320} on the determination of the solar apex coordinates. The last row corresponds to the sample of 45 803 stars in which only the coefficients r_{001}, r_{110}, r_{111}, and r_{320} from Table 3 were taken into account for the distribution of stars in space. The dimensions are pc (columns 1–8) and deg (columns 9 and 10)

r_{001}	r_{110}	r_{111}	r_{320}	r_{min}	$\hat{r}$	r_{max}	$r_{max}-r_{min}$	ΔL	ΔB
1000	200	−200	200	061	282	503	442	2.1	−1.4
1000	200	−200	300	007	282	557	550	2.1	−2.0
2000	200	−200	100	393	564	737	344	2.1	−0.8
2000	300	−300	300	232	564	896	664	3.1	−2.1
2000	500	−500	500	011	564	1188	1177	5.1	−3.5
688	46	−40	0	164	194	223	59	0.5	0.0

Table 7. Influence of the coefficients r_{001} and r_{201} on the shift in the Oort parameter $\Delta = M_{21}^{+} - 15$ km s^{-1}. The last row corresponds to the sample of 45 803 stars. The dimensions are pc (columns 1–6) and km s^{-1} kpc^{-1} (column 7)

r_{001}	r_{201}	r_{min}	$\hat{r}$	r_{max}	$r_{max}-r_{min}$	Δ
1000	300	188	282	469	281	−2.8
1000	500	126	282	594	468	−4.7
2000	500	408	564	876	468	−2.5
2000	1000	252	564	1188	936	−4.7
688	−69.6	151	194	216	65	0.97

$A = M_{12}^{+}$. For this purpose, we will again produce a model catalog of radial velocities using Eq. (4) with parameters $U = 10$, $V = 15$, $W = 7$ km s^{-1}, and $M_{12}^{+} = 15$ km s^{-1} kpc^{-1}. All of the remaining parameters are assumed to be zero. The results of our calculations in the absence of a noise component are given in Table 7. If the noise component simulating the "cosmic dispersion" with the parameters used to obtain the results in Table 6 is introduced into the calculated radial velocities as before, then we can find that the rms error of the shift ΔM_{12}^{+} decreases from 1.2 to 0.6 km s^{-1} kpc^{-1} as r_{001} changes from 1000 to 2000 pc. Analyzing Table 4, it can be assumed that the shifts in the Oort parameter $A = M_{12}^{+}$ are determined mainly by the zonal coefficients r_{001} and r_{201}, i.e., by the range of distances and the degree of flattening of the spatial distribution of stars to the Galactic plane. It can be shown that the estimate of the relative error in the Oort parameter $A = M_{12}^{+}$ is specified by the formula

$$\frac{M_{12}^{+} - (M_{12}^{+})_0}{(M_{12}^{+})_0} = -0.64\frac{r_{201}}{r_{001}}. \qquad (15)$$

It can thus be seen that significant (at a level of 20–30%) shifts in the Oort parameter may be expected in the case of a strong flattening of the system of stars to the Galactic plane corresponding to r_{201}/r_{001} at a level of 0.3–0.5. This explains why this shift for the sample of 45 803 stars turned out to be insignificant ($r_{201}/r_{001} = -0.1$).

Table 8. Values of the beyond-the-model coefficient v_{310}. The parameters of the samples from the CRVAD-2 catalog: N is the number of stars, r_{min} is the minimum distance to the sample stars (pc), $\hat{r}$ is the mean distance to the sample stars (pc), r_{max} is the maximum distance to the sample stars (pc); v_{310} is the beyond-the-model harmonic derived from the real radial velocities (km s^{-1})

Sample	N	r_{min}	$\hat{r}$	r_{max}	v_{310}
I	45803	90	194	450	-3.8 ± 0.6
II	26573	148	241	509	-5.0 ± 0.8
III	12043	214	318	559	-9.4 ± 1.8
IV	3080	302	344	400	-11.0 ± 3.8

BEYOND-THE-MODEL HARMONICS

It has been said above that the coefficients of the decomposition of radial velocities to $n \leq 2$ determine the parameters in Eq. (4). This allows the coefficients of the decomposition of radial velocities to be separated into two sets. It is appropriate to call the first set "model coefficients", because they allow the parameters of the Ogorodnikov–Milne model (4) to be determined. "Beyond-the-model" coefficients corresponding to the kinematic effects that do not enter into the standard model constitute the second set. Examples of obtaining these two types of coefficients are given in Vityazev and Tsvetkov (2009). An important result obtained in this paper is that the coefficient v_{310} always turns out to be significant among the coefficients of the higher harmonics in the decomposition of radial velocities. These values were derived from the OSACA catalog of radial velocities (Bobylev et al. 2006). In this case, not the radial velocities themselves but the ratios V_r/r were subjected to a decomposition. To confirm these results, we drew several samples from the CRVAD-2 catalog and performed spherical harmonic decompositions of the radial velocities themselves for them. Among the beyond-the-model coefficients at $n > 2$, only the coefficient v_{310} turned out to be significant as before. The values obtained for this coefficient and the parameters of the samples of stars are given in Table 8. It can be seen from this table that the absolute value of the coefficient v_{310} increases with increasing mean distance to the stars. It should be noted that a similar picture was also obtained from the OSACA catalog: $v_{310} = -5.5 \pm 2.2$ km s^{-1} for a mean distance of 0.243 kpc and $v_{310} = -14.5 \pm 4.7$ km s^{-1} for a mean distance of 0.348 kpc, in good agreement with the results given in Table 8. The reason for the existence of this harmonic in the radial velocities requires a further study.

CONCLUSIONS

The main goal of this paper is to construct a method for analyzing the radial velocities using spherical harmonics. This approach does not depend on the specific kinematic model and allows both low-frequency and high-frequency kinematic radial velocity components to be analyzed. We considered two cases. In the first case, we assume the sample of stars being studied to be at approximately the same heliocentric distance; in the second case, we take into account the possible systematic variations of distances with coordinates on the celestial sphere that, in turn, are also modeled by a linear combination of spherical harmonics. Theoretical relations showing how the coefficients of the decomposition of distances affect the coefficients of the decomposition of the radial velocities themselves were derived. We established that the coefficients r_{nkp} odd in index n affect only the determination of the solar motion components and the calculation of the solar apex coordinates. We showed that the larger the mean distance to the sample of stars being analyzed, the greater the shift in the solar apex coordinates. The coefficients r_{nkp} with even values of the index n were shown to affect only the deformation tensor elements, with the shifts in the Oort parameter $A = M^+_{12}$ being determined mainly by the coefficients r_{001} and r_{201}, i.e., by the range of distances to the stars and the degree of flattening of their spatial distribution to the Galactic plane. These results should be compared with the results of a kinematic analysis of stellar proper motions. Previously (Vityazev and Tsvetkov 2013, 2014), we studied the influence of the systematic variations of stellar parallaxes with coordinates on the determination of the velocity field parameters from stellar proper motions. Just as for the analysis of radial velocities, we used the same approach for the analysis of stellar proper motions: the representation of their annual parallaxes by a linear combination of spherical harmonics. In both cases, we derived formulas showing how the coefficients of the corresponding decompositions distort the parameters of the kinematic Ogorodnikov–Milne model. These formulas suggest that all Ogorodnikov–Milne model parameters are influenced by the systematic variations of distances (parallaxes) with coordinates, but the degree of this influence is different for nearby and distant stars. Indeed, as can be seen from Eqs. (2)–(4), the influence of the systematic variations of distances on the proper motions decreases when passing from nearby stars to distant ones, while the reverse is true for the radial velocities. From this viewpoint, to remove the influence of the systematic variations of stellar parallaxes over the sky, it is desirable to take distant stars ($r > 1$ kpc) when analyzing the proper motions and, conversely, nearby stars ($r < 1$ kpc) when analyzing the radial velocities.

Note also that, as in the case of stellar proper motions, a kinematic component that has no explanation in terms of the three-dimensional Ogorodnikov–Milne model is detected in the radial velocities. We first detected this effect when analyzing the radial velocities from the OSACA catalog (Vityazev and Tsvetkov 2009). In this paper, we confirmed it when analyzing various samples of stars from the CRVAD-2 catalog of radial velocities.

ACKNOWLEDGMENTS

This work was supported by grant no. 6.0.161.2010 of the St. Petersburg State University.

REFERENCES

1. G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1966; Atomizdat, Moscow, 1970).
2. V. V. Bobylev, G. A. Gontcharov, and A. T. Bajkova, Astron. Rep. **50**, 733 (2006).
3. P. N. Fedorov, A. A. Myznikov, and V. S. Akhmetov, Mon. Not. R. Astron. Soc. **393**, 133 (2009).
4. N. V. Kharchenko, R.-D. Scholz, A. E. Piskunov, S. Roeser, and E. Schilbach, Astron. Nachr. **328**, 889 (2007).
5. F. van Leeuwen, *Hipparcos, the New Reduction of the Raw Data,* Space Science Library (Springer, Berin, 2007), Vol. 350.
6. M. Miyamoto, M. Soma, and M. Yokoshima, Astron. J. **105**, 2138 (1993).
7. B. A. du Mont, Astron. Astrophys. **61**, 127 (1977).
8. K. F. Ogorodnikov, *Dynamics of Stellar Systems* (Fizmatgiz, Moscow, 1965) [in Russian].
9. R. P. Olling and W. Dehnen, Astrophys. J. **599**, 275 (2003).
10. M. A. C. Perriman et al., *The Hipparcos and Tycho Catalogues,* Vols. 1–17, ESA SP-1200 (1997).
11. S. Roeser, M. Demleitner, and E. Schilbach, Astron. J. **139**, 2440 (2010).
12. V. V. Vityazev and A. S. Tsvetkov, Astron. Lett. **35**, 100 (2009).
13. V. V. Vityazev and A. S. Tsvetkov, Astron. Lett. **38**, 411 (2012).
14. V. V. Vityazev and A. S. Tsvetkov, Vestn. SPbGU, Ser. 1, No. 2, 138 (2013).
15. V. V. Vityazev and A. S. Tsvetkov, Astron. Nachr. **334**, No. 8 (2013).
16. V. V. Vityazev and A. S. Tsvetkov, Astron. Lett. **40**, 46 (2014).
17. N. Zacharias, C. T. Finch, T. M. Girard, A. Henden, J. L. Bartlett, D. G. Monet, and M. I. Zacharias, Astron. J. **145**, 44 (2013).

Translated by V. Astakhov