ISSN 1063-7737, Astronomy Letters, 2015, Vol. 41, No. 7, pp. 317–333. © Pleiades Publishing, Inc., 2015.
Original Russian Text © V.V. Vityazev, A.S. Tsvetkov, 2015, published in Pis'ma v Astronomicheskiĭ Zhurnal, 2015, Vol. 41, No. 7, pp. 350–366.

Systematic Differences between the Positions and Proper Motions of Stars from the PPMXL and UCAC4 Catalogs

V. V. Vityazev* **and A. S. Tsvetkov****

St. Petersburg State University, Bibliotechnaya pl. 2, St. Petersburg, 198504 Russia

Received February 10, 2015

Abstract—The PPMXL and UCAC4 catalogs are compared by representing the differences between the positions and proper motions of stars as decompositions into a set of orthogonal vector harmonics with allowance made for the magnitude equation. A list of 41 316 676 common stars has been compiled using the star identification procedure in the J band (2MASS photometric system). The mean differences between the stellar positions and proper motions have been referred to the centers of 1200 HealPix pixels on the sphere. These data have been generated in the equatorial coordinate system for the stars belonging to 12 J magnitude bins with a width of 0.5^m for mean values from $10\overset{m}{.}25$ to $15\overset{m}{.}75$. For each sample of stars, the differences have been approximated by vector spherical harmonics. A new statistical criterion that is oriented to using HealPix data pixelization and that allows the significance of all the accessible harmonics to be determined is proposed to extract the signal from noise. An analytical method that includes the effects dependent on the magnitude of the stars has been proposed for the first time in the vector spherical harmonics technique, i.e., a new model of systematic differences based on a system of basis functions that are the products of vector spherical harmonics and Legendre polynomials has been generated. The influence of the magnitude equation on the determination of the mutual orientation and rotation of the PPMXL and UCAC4 reference frames has been studied. It has been established that the extreme systematic differences do not exceed in absolute value 20 mas and 4 mas yr^{-1} for the positions and proper motions, respectively. The largest differences between the PPMXL and UCAC4 catalogs are shown to be explained by their random rather than systematic errors.

DOI: 10.1134/S1063773715070063

Keywords: *astrometry, systematic differences, star catalogs, spherical harmonics.*

INTRODUCTION

In anticipation of catalogs within the framework of the GAIA project, the PPMXL (Roeser et al. 2010) and UCAC4 (Zacharias et al. 2013) all-sky astrometric catalogs provide a basis for performing various astronomical studies. The PPMXL catalog contains information about the ICRS positions and proper motions of ∼900 million stars down to magnitude $V = 20$ with complete sky coverage. The mean errors of the proper motions lie within the range from 4 to 10 mas yr^{-1}, while the positional accuracy at epoch 2000.0 is estimated to be from 80 to 120 mas for 410 million objects for which the positions in the 2MASS catalog (Skrutskie et al. 2006) are known. For the remaining stars, the positional accuracy varies between 150 and 300 mas.

The UCAC4 catalog contains 113 million stars from magnitude 8 to 16 in a nonstandard photometric band between V and R. It also covers the entire sky. The positional accuracy at the mean epoch is estimated to be within the range 15–100 mas, while the formal errors of the proper motions are within the range 1–10 mas yr^{-1}. The systematic errors of the proper motions lie within the range 1–4 mas yr^{-1}. The catalog was constructed in the ICRS and is deemed complete down to $R = 16$. The UCAC4 is the last catalog in the UCAC (USNO CCD Astrograph Catalog) project. No photographic observations were used in this project, because all measurements were made between 1998 and 2004 using only CCD detectors.

At present, these catalogs are widely used as the reference frames extending the ICRS in the optical range to hundreds of millions of stars. In accordance with the requirements of astrometry, it is necessary to have the opportunity to pass from the system of one catalog to the system of another catalog. The authors of the UCAC4 catalog (Zacharias et al. 2013) compared the proper motions of stars from the PPMXL and UCAC4 catalogs in a narrow RA zone from 6.0 to 6.1 h in the declination range from −60° to −30°.

*E-mail: vityazev@list.ru
**E-mail: a.s.tsvetkov@inbox.ru

Farnocchia et al. (2015) obtained the corrections to the UCAC4 stellar positions and proper motions in the form of differences PPMXL−UCAC4 referred to the centers of the pixels produced by the HealPix partition of the celestial sphere (Gorski et al. 2005). Such an approach suggests using numerical interpolation to calculate the differences for a specific point on the sphere. In addition, the dependence of these differences on the magnitude of stars was disregarded in the presented data, and no smoothing over the right ascension and declination was performed to reduce the random errors.

A proper solution of the problem of comparing catalogs (Bien et al. 1978; Mignard and Froeschle 2000) suggests representing the systematic differences by the systems of orthogonal harmonics describing their dependence on the coordinates and magnitudes of stars. To all appearances, the PPMXL and UCAC4 catalogs have not yet subjected to comparison with such a degree of completeness, and this paper is devoted to remedying this shortcoming. It is based on pre-pixelization of the individual differences between stars from different magnitude groups followed by their approximation by vector spherical harmonics. In contrast to previous similar works, here we propose a new statistical criterion that allows one to estimate the significance of all the harmonics that can be calculated on the chosen HealPix pixelization scheme. Normalized Legendre polynomials are used to approximate the decomposition coefficients derived from groups of stars with different magnitudes. The constructed models of systematic differences are used to analyze the systematic differences as functions of three variables (α, δ, m).

MODELING THE SYSTEMATIC DIFFERENCES. THE SCALAR CASE

Brosche (1966) was the first to represent the systematic differences between the positions and proper motions of stars by scalar spherical harmonics. In his method, the model of systematic position differences appeared as follows:

$$\Delta\alpha\cos\delta = \sum_{nkp} v^{\alpha}_{nkp} K_{nkp}(\alpha,\delta), \qquad (1)$$

$$\Delta\delta = \sum_{nkp} v^{\delta}_{nkp} K_{nkp}(\alpha,\delta), \qquad (2)$$

where v^{α}_{nkp} and v^{δ}_{nkp} are the coefficients of the decomposition of data into a system of spherical harmonics defined by the expression

$$K_{nkp}(\alpha,\delta) \qquad (3)$$

$$= R_{nk}\begin{cases} P_{n,0}(\delta), & k=0, \quad p=1, \\ P_{nk}(\delta)\sin k\alpha, & k\neq 0, \quad p=0, \\ P_{nk}(\delta)\cos k\alpha, & k\neq 0, \quad p=1; \end{cases}$$

$$R_{nk} = \sqrt{\frac{2n+1}{4\pi}}\begin{cases} \sqrt{\frac{2(n-k)!}{(n+k)!}}, & k>0, \\ 1, \quad k=0, \end{cases} \qquad (4)$$

where α and δ are the right ascension (longitude) and declination (latitude) of a point on the sphere, respectively ($0\le\alpha\le 2\pi$; $-\pi/2\le\delta\le\pi/2$); $P_{nk}(\delta)$ are Legendre polynomials (at $k=0$) and associated Legendre functions (at $k>0$), which can be calculated using the following recurrence relations:

$$P_{nk}(\delta) = \sin\delta\frac{2n-1}{n-k}P_{n-1,k}(\delta) \qquad (5)$$
$$-\frac{n+k-1}{n-k}P_{n-2,k}(\delta),$$
$$k=0,1,\dots n=k+2,k+3,\dots,$$
$$P_{kk}(\delta) = \frac{(2k)!}{2^k k!}\cos^k\delta,$$
$$P_{k+1,k}(\delta) = \frac{(2k+2)!}{2^{k+1}(k+1)!}\cos^k\delta\sin\delta.$$

When working with spherical harmonics, one index j is often used for the convenience of their numbering, with

$$j = n^2 + 2k + p - 1. \qquad (6)$$

The introduced functions satisfy the relation

$$\iint_{\Omega} (K_i\cdot K_j)\,d\omega = \begin{cases} 0, & i\neq j, \\ 1, & i=j. \end{cases} \qquad (7)$$

In other words, the set of functions K_{nkp} forms an orthonormal system of functions on the sphere.

USING VECTOR SPHERICAL HARMONICS TO REPRESENT THE SYSTEMATIC DIFFERENCES BETWEEN THE POSITIONS AND PROPER MOTIONS OF STARS

The systematic differences between the positions and proper motions are the components of some vector field. Therefore, it seems appropriate to use the technique of decomposing this field into a system of vector spherical harmonics (below referred to as VSHs) to study the systematic differences. Note that VSHs were first used by Mignard and Morando (1990) and Mignard and Froeschle (2000) in astrometric problems related to the comparison of catalogs to represent the systematic differences between Hipparcos and FK5. A further development of this technique aimed at its application in the GAIA project can be found in Mignard and Klioner (2012). In this paper, we will use the VSH apparatus in the form in which it was applied in our previous papers on a kinematic analysis of stellar proper motions (Vityazev and Tsvetkov 2013, 2014).

Consider a system of mutually orthogonal unit vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\delta$, respectively, in the directions of change in right ascension and declination in a plane tangential to the sphere. Using the definitions of VSHs in Arfken (1966), let us introduce toroidal, $\mathbf{T}_{nkp}$, and spheroidal, $\mathbf{S}_{nkp}$, VSHs via the relations

$$\mathbf{T}_{nkp}(\alpha,\delta) = \frac{1}{\sqrt{n(n+1)}} \times \left(\frac{\partial K_{nkp}(\alpha,\delta)}{\partial\delta}\mathbf{e}_\alpha - \frac{1}{\cos\delta}\frac{\partial K_{nkp'}(\alpha,\delta)}{\partial\alpha}\mathbf{e}_\delta\right), \tag{8}$$

$$\mathbf{S}_{nkp}(\alpha,\delta) = \frac{1}{\sqrt{n(n+1)}} \times \left(\frac{1}{\cos\delta}\frac{\partial K_{nkp}(\alpha,\delta)}{\partial\alpha}\mathbf{e}_\alpha + \frac{\partial K_{nkp}(\alpha,\delta)}{\partial\delta}\mathbf{e}_\delta\right), \tag{9}$$

where $K_{nkp}(\alpha,\delta)$ are the scalar spherical harmonics specified by Eq. (3).

Denote the components of the unit vector $\mathbf{e}_\alpha$ as T^α_{nkp} and S^α_{nkp} and the components of the unit vector $\mathbf{e}_\delta$ as T^δ_{nkp} and S^b_{nkp}, respectively:

$$\mathbf{T}_{nkp} = T^\alpha_{nkp}\mathbf{e}_\alpha + T^\delta_{nkp}\mathbf{e}_\delta, \tag{10}$$

$$\mathbf{S}_{nkp} = S^\alpha_{nkp}\mathbf{e}_\alpha + S^\delta_{nkp}\mathbf{e}_\delta. \tag{11}$$

Given that $P_{n,k+1}(b) = 0$ at $n < k+1$, these components are defined as

$$T^\alpha_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \tag{12}$$

$$\times \begin{cases} P_{n,1}(\delta), \quad k=0, \quad p=1, \\ (-k\tan\delta P_{nk}(\delta) + P_{n,k+1}(\delta))\sin k\alpha, \\ k\neq 0, \quad p=0, \\ (-k\tan\delta P_{nk}(\delta) + P_{n,k+1}(\delta))\cos k\alpha, \\ k\neq 0, \quad p=1; \end{cases} \tag{13}$$

$$T^\delta_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \tag{14}$$

$$\times \begin{cases} 0, \quad k=0, \quad p=1, \\ -\frac{k}{\cos\delta}P_{nk}(\delta)\cos k\alpha, \quad k\neq 0, \quad p=0, \\ +\frac{k}{\cos\delta}P_{nk}(\delta)\sin k\alpha, \quad k\neq 0, \quad p=1; \end{cases} \tag{15}$$

$$S^\alpha_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \tag{16}$$

$$\times \begin{cases} 0, \quad k=0, \quad p=1, \\ +\frac{k}{\cos\delta}P_{nk}(\delta)\cos k\alpha, \quad k\neq 0, \quad p=0, \\ -\frac{k}{\cos\delta}P_{nk}(\delta)\sin k\alpha, \quad k\neq 0, \quad p=1; \end{cases} \tag{17}$$

$$S^\delta_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \tag{18}$$

$$\times \begin{cases} P_{n,1}(\delta), \quad k=0, \quad p=1, \\ (-k\tan\delta P_{nk}(\delta) + P_{n,k+1}(\delta))\sin k\alpha, \\ k\neq 0, \quad p=0, \\ (-k\tan\delta P_{nk}(\delta) + P_{n,k+1}(\delta))\cos k\alpha, \\ k\neq 0, \quad p=1. \end{cases}$$

The introduced functions satisfy the relations

$$\iint_\Omega (\mathbf{T}_i\cdot\mathbf{T}_j)\,d\omega = \iint_\Omega (\mathbf{S}_i\cdot\mathbf{S}_j)\,d\omega = \begin{cases} 0, \quad i\neq j, \\ 1, \quad i=j, \end{cases} \tag{19}$$

$$\iint_\Omega (\mathbf{S}_i\cdot\mathbf{T}_j)\,d\omega = 0, \quad \forall i,j. \tag{20}$$

In other words, the set of functions $\mathbf{T}_{nkp}$ and $\mathbf{S}_{nkp}$ forms a orthonormal system of functions on the sphere.

VSH DECOMPOSITION OF THE STELLAR VELOCITY FIELD

Consider the actual field of systematic stellar position differences on the celestial sphere:

$$\Delta\mathbf{F}(\alpha,\delta) = \Delta\alpha\cos\delta\,\mathbf{e}_\alpha + \Delta\delta\,\mathbf{e}_\delta. \tag{21}$$

We will also use a similar expression to represent the systematic differences between the stellar proper motions.

Using the system of VSHs defined above, we can decompose the field of differences as

$$\Delta\mathbf{F}(\alpha,\delta) = \sum_{nkp} t_{nkp}\mathbf{T}_{nkp}(\alpha,\delta) + \sum_{nkp} s_{nkp}\mathbf{S}(\alpha,\delta)_{nkp}(\alpha,\delta), \tag{22}$$

where, since the basis is orthonormal, the decomposition coefficients t_{nkp} and s_{nkp} can be calculated from the formulas

$$t_{nkp} = \iint_\Omega (\Delta\mathbf{F}\cdot\mathbf{T}_{nkp})\,d\omega \tag{23}$$

$$= \int\limits_{0}^{2\pi} d\alpha \int\limits_{-\pi/2}^{+\pi/2} \left(\Delta\alpha \cos\delta T^{\alpha}_{nkp} + \Delta\delta\, T^{\delta}_{nkp}\right) \cos\delta d\delta,$$

$$s_{nkp} = \iint\limits_{\Omega} \left(\Delta\mathbf{F} \cdot \mathbf{S}_{nkp}\right) d\omega \tag{24}$$

$$= \int\limits_{0}^{2\pi} d\alpha \int\limits_{-\pi/2}^{+\pi/2} \left(\Delta\alpha \cos\delta S^{\alpha}_{nkp} + \Delta\delta\, S^{\delta}_{nkp}\right) \cos\delta d\delta.$$

When catalogs containing hundreds of millions of stars are compared, it is appropriate to use data prepixelization. We will assume equal-area pixels to be constructed on the sphere according to the HealPix scheme (Gorski et al. 2005). In this scheme, the number $N_{\rm pix}$ is the key parameter (resolution parameter) defining the partition of the sphere into equal pixels. The total number of pixels is $N = 12N^2_{\rm pix}$. The entire sphere is divided by two parallels with declinations $\pm\arcsin(2/3)$ into three parts, the equatorial and two polar ones. $N_{\rm pix} - 1$ parallels is chosen in each of the polar zones; the number of parallels in the equatorial zone is $(2N_{\rm pix} + 1)$. The centers of $4N_{\rm pix}$ pixels lie on each parallel of the equatorial region. Two parallels closest to the poles always contain four pixels, while the number of pixels on each parallel increases by one when moving from the poles to the equator in the polar zones. The pixels are numbered $j = 0, 1, \ldots, N - 1$ along the parallels from north to south.

To calculate the coefficients of the VSH decomposition of the systematic differences, we now have the following formulas instead of Eqs. (23) and (24):

$$t_{nkp} = \frac{4\pi}{N} \sum_{j=0}^{N-1} \Delta\mathbf{F}(\alpha_j, \delta_j)\mathbf{T}_{n,k1,p}(\alpha_j, \delta_j), \tag{25}$$

$$s_{nkp} = \frac{4\pi}{N} \sum_{j=0}^{N-1} \Delta\mathbf{F}(\alpha_j, \delta_j)\mathbf{S}_{n,k1,p}(\alpha_j, \delta_j). \tag{26}$$

Calculating the toroidal and spheroidal coefficients is the main goal of the problem of representing the systematic differences between the positions and proper motions of stars from two catalogs by VSHs.

STATISTICAL CRITERIA FOR DETERMINING SIGNIFICANT TERMS IN THE VSH DECOMPOSITION OF SYSTEMATIC DIFFERENCES

In the analytical method (Brosche 1966; Bien et al. 1978), the highest-order decomposition term fixes the boundary between the systematic and random components. In the scalar case, to find the highest-order decomposition term, Fisher's F-test is applied to test the hypothesis that the systematic part in each g decomposition step is limited by number $g - 1$; therefore, the true values of the coefficients $a_g, a_{g+1}, \ldots$ are zero. If Fisher's test is fulfilled with a specified probability, then the hypothesis is deemed admissible. However, it is not accepted automatically, because significant decomposition terms can be found for higher-order decomposition terms. Taking this into account, the decompositions are continued to some number of harmonics ($\simeq$50), and if there is no significant harmonic among them, then the hypothesis is accepted with the initial g. Obviously, the absence of a well-defined criterion for determining the highest-order decomposition term ($\simeq$50) breaks the strictness of this method.

When using VSHs, Mignard and Klioner (2012) proposed testing the significance of not each decomposition term but the whole set of harmonics with the same index n. In their opinion, this approach is appropriate, because the established statistical significance of such a set of harmonics is invariant with respect to rotation of the initial coordinate system. In other words, if a group of terms in the decomposition of the systematic differences with fixed index n is statistically significant in the equatorial coordinate system, then it will also have the same significance level in the Galactic coordinate system.

Nevertheless, it should be said that this approach can cause the general set of decomposition terms to be "littered," because each set of harmonics with fixed index n consists of $2n + 1$ terms, while a high statistical significance can be provided only by one harmonic. In contrast to this, we can propose a method that allows the signal (not noise) components of the decomposition to be extracted with a specified probability among all the admissible indices k and n for the pixelization scheme used, following which the numerical values of the significant decomposition coefficients are immediately determined from a single application of the least-squares method (LSM). This approach is analogous to the spectral analysis of evenly spaced time series, where the significance of the periodogram peaks is determined for all its peaks in the frequency range specified by the sampling step. In our case, the squares of the coefficients s_{nkp} and t_{nkp} may be considered as the periodogram values, and the determination of their significance is based on the fact that the coefficients s_{nkp} and t_{nkp} for normally distributed centered noise with variance $\sigma_0^2 = 1$ are normally distributed random variables with zero mean and unit variance. Consequently, the squares of the amplitudes s^2_{nkp} and t^2_{nkp} are random variables distributed according to the chi-square law with one degree of freedom. On this basis, we can estimate the probability q that s^2_{nkp} and t^2_{nkp} exceed a threshold X:

$$q = \int\limits_X^\infty p_k(x)dx, \tag{27}$$

where $p_k(x)$ is the density of the χ^2 distribution with k degrees of freedom. Hence it follows that the determination of the significance of each harmonic is based on testing the a priori hypothesis that the initial data are discrete centered noise with unit variance. This hypothesis is tested for each harmonic and is rejected with a probability $p = 1 - q$ if the square of the decomposition coefficient of the centered normalized data sequence exceeds the detection threshold X determined from Eq. (27).

As has been said above, to determine the significance of the decomposition coefficients, it is necessary to exhaust all of the harmonics that can be calculated on a chosen grid of points. This requirement is reduced to establishing the boundary values of the indices k and n. To choose the largest $k_{\max}$, we can use the fact that the orthogonality condition on the HealPix grid with the key parameter N_{pix} is violated for the products of the harmonics $\mathbf{T}_{k1,k1,p}(l_j, b_j)\mathbf{T}_{k2,k2,p}(l_j, b_j)$ and $\mathbf{S}_{k1,k1,p}(l_j, b_j)\mathbf{S}_{k2,k2,p}(l_j, b_j)$ when the following relation holds:

$$k1 + k2 = 8rN_{\text{pix}}, \quad r = 1, 2, \ldots . \tag{28}$$

This means that $k = k_{\max} = 4N_{\text{pix}}$ is the boundary value in the sense that each VSH with indices $n = k > 4N_{\text{pix}}$ will give a false decomposition coefficient with indices $n = k < 4N_{\text{pix}}$. Therefore, the harmonics should be exhausted in index k for $k = 0, 1, \ldots, 4N_{\text{pix}} - 1$. Continuing the analogy with time series, it can be said that $4N_{\text{pix}}$ in our problem is an analog of the Nyquist frequency.

A constraint on the index n can be derived from the condition that the sought-for decomposition coefficients are obtained with a specified accuracy. Since there exists a constraint on the accuracy of calculating the squares of the norms of the basis functions in index n on a discrete grid of HealPix pixel centers, the limiting value of our series $n = k, k+1, \ldots, n_{\max}$ for each admissible index k is determined from the condition that a specified accuracy (for example, one percent) of calculating the squares of the norms of the basis functions breaks down:

$$\left|1 - \frac{4\pi}{N}\sum_{j=0}^{N-1}\mathbf{S}(n_{\max}, k, p, \alpha_j, \delta_j) \times \mathbf{S}(n_{\max}, k, p, \alpha_j, \delta_j)\right| > 0.01, \quad p = 0, 1, \tag{29}$$

$$\left|1 - \frac{4\pi}{N}\sum_{j=0}^{N-1}\mathbf{T}(n_{\max}, k, p, \alpha_j, \delta_j) \times \mathbf{T}(n_{\max}, k, p, \alpha_j, \delta_j)\right| > 0.01, \quad p = 0, 1. \tag{30}$$

Thus, within the ranges of admissible indices k and n, the inequalities

$$\frac{N}{4\pi}\tilde{s}^2_{nkp} > X, \quad \frac{N}{4\pi}\tilde{t}^2_{nkp} > X \tag{31}$$

suggest that the coefficients with indices n, k, and p are determined with a specified probability $p = 1 - q$ by the presence of a corresponding harmonic rather than noise.

In these formulas, $\tilde{s}_{nkp}$ and $\tilde{t}_{nkp}$ are calculated from Eqs. (25) and (26) based on the normalized initial data

$$\Delta\tilde{\mathbf{F}}(\alpha, \delta) = \frac{\Delta\alpha\cos\delta - \langle\Delta\alpha\cos\delta\rangle}{\sigma_\alpha}\,\mathbf{e}_\alpha + \frac{\Delta\delta - \langle\Delta\delta\rangle}{\sigma_\delta}\,\mathbf{e}_\delta, \tag{32}$$

where the means and variances of the initial differences were obtained as follows:

$$\langle\Delta\alpha\cos\delta\rangle = \frac{1}{N}\sum_{j=0}^{N-1}(\Delta\alpha\cos\delta_j), \tag{33}$$

$$\sigma^2_\alpha = \frac{1}{N-1}\sum_{j=0}^{N-1}(\Delta\alpha\cos\delta_j - \langle\Delta\alpha\cos b\rangle)^2, \tag{34}$$

$$\langle\Delta\delta\rangle = \frac{1}{N}\sum_{j=0}^{N}(\Delta\delta_j), \tag{35}$$

$$\sigma^2_\delta = \frac{1}{N-1}\sum_{j=0}^{N-1}(\Delta\delta_j - \langle\Delta\delta\rangle)^2. \tag{36}$$

Centering and normalizing the differences $\Delta\alpha\cos\delta_j$ and $\Delta\delta_j$ allow the hypothesis about the chi-square distribution of the sought-for coefficients with one degree of freedom to be used. However, for the coefficients $t_{n,0,1}$ and $s_{n,0,1}$ at odd indices n, not the coefficients themselves but only the following values are tested:

$$\frac{N}{4\pi}(\tilde{s}_{nkp} - \hat{s}_{nkp})^2 > X, \quad n = 1, 3, 5, \ldots, \qquad (37)$$

$$\frac{N}{4\pi}(\tilde{t}_{nkp} - \hat{t}_{nkp})^2 > X, \quad n = 1, 3, 5, \ldots, \qquad (38)$$

where $\hat{s}_{nkp}$ and $\hat{t}_{nkp}$ correspond to (25) and (26), with

$$\Delta\tilde{\mathbf{F}}(\alpha, \delta) = \frac{\langle \Delta\alpha \cos\delta \rangle}{\sigma_\alpha}\, \mathbf{e}_\alpha + \frac{\langle \Delta\delta \rangle}{\sigma_\delta}\, \mathbf{e}_\delta. \qquad (39)$$

Obviously, in this case, we can judge only the probability of a deviation of the sought-for coefficient from its value corresponding to the case where the individual differences do not depend on the coordinates.

Thus, after the selection of the indices n, k, and p for which the decomposition coefficients are significant, the numerical values of the coefficients themselves and their root-mean-square (rms) errors are determined by the LSM from the selected set of functions. Since the significance test based on the chi-square distribution cannot be applied to the zonal functions with odd indices n, the zonal functions $S_{n,0,1}$ and $T_{n,0,1}$ at $n = 1, 3, 5, \ldots, n_{\max}$, where $n_{\max}$ is assigned from conditions (29) and (30) at $k = 0$ and $p = 1$, should always be included in the set of functions for the LSM solution.

DESCRIPTION OF THE MAGNITUDE EQUATION

In analyzing astrometric catalogs, the dependence of systematic differences on the magnitude of stars in a particular photometric band is traditionally called the "magnitude equation." Already in his pioneering paper, Brosche (1966) found that the decomposition coefficients of the systematic differences between the FK4 and GC catalogs differed significantly when they were determined from bright and faint stars. To take this dependence into account, Brosche proposed to use the products of scalar spherical harmonics and Legendre polynomials as basis functions. Bien et al. (1978) proposed a modification of this approach in which Legendre–Fourier–Hermit functions were used. This system of functions, in which the magnitude equation was taken into account using Hermit polynomials, became a basis for the standard method of a separate comparison of the systems of right ascensions and declinations for astrometric catalogs with the application of scalar basis functions. It can be shown that, in the scalar case, this approach is based on the following model of the representation of systematic differences:

$$f(\alpha, \delta, m) = \sum_j b_j(m) Z_j(\alpha, \delta), \qquad (40)$$

where it is explicitly specified that the coefficients of the decomposition of systematic differences into basis functions $Z_j(\alpha, \delta)$ are functions of the magnitude.

The described approach to take into account the magnitude equation using scalar harmonics can also be easily extended to VSHs. In this case, an analog of model (40) is the expression

$$\Delta\mathbf{F}(\alpha, \delta, m) = \sum_{nkp} t_{nkp}(m)\mathbf{T}_{nkp}(\alpha, \delta) + \sum_{nkp} s_{nkp}(m)\mathbf{S}_{nkp}(\alpha, \delta), \qquad (41)$$

where the coefficients $t_{nkp}(m)$ and $s_{nkp}(m)$ are functions of the magnitude. These coefficients are approximated using appropriate polynomials $Q_r(m)$ by expressions of the form

$$t_{nkp}(m) = \sum_r t_{nkpr} Q_r(m), \qquad (42)$$
$$s_{nkp}(m) = \sum_r s_{nkpr} Q_r(m).$$

Substituting (42) into (41) gives the final form of the model of systematic differences

$$\Delta\mathbf{F}(\alpha, \delta, m) = \sum_{nkpr} t_{nkpr}\mathbf{T}_{nkp}(\alpha, \delta)Q_r(m) + \sum_{nkpr} s_{nkpr}\mathbf{S}_{nkp}(\alpha, \delta)Q_r(m). \qquad (43)$$

Obviously, there exist two approaches to consider the magnitude equation when the systematic differences are approximated by orthogonal functions. In the first method, model (41) is used to first determine the coefficients $t_{nkp}(m)$ and $s_{nkp}(m)$ from the stars belonging to small magnitude bins, and then the derived decomposition coefficients referred to the mean values of the magnitude bins are approximated by expressions of form (42). This approach is possible if all-sky catalogs are available, when samples containing a sufficiently large number of stars with approximately the same magnitude can be produced. The second method is based on the direct solution of conditional equations of form (43) without any pre-averaging of the systematic differences over the magnitude bins. It is quite clear that it is appropriate to apply this approach when the number of stars in the comparison catalogs is small. In the scalar case, the standard method using Legendre–Fourier–Hermit functions was based precisely on this approach. The coefficients t_{nkpr} and s_{nkpr} have not yet been determined for VSHs.

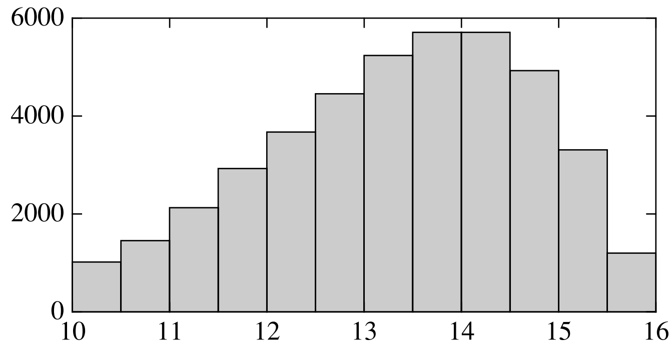

Fig. 1. Magnitude distribution of stars from the sample of 41 316 676 stars. The J magnitudes and the number of stars (in thousands) are along the horizontal and vertical axes, respectively.

NUMERICAL RESULTS

In our work, we used a method based on VSHs with allowance made for the magnitude equation to study the systematic differences between the PPMXL and UCAC4 catalogs. The systematic differences between the proper motions were obtained in three steps.

In the first step, we partitioned the sphere by the HealPix method into 1200 pixels with an area of 34.4 sq. deg. Using the star identification procedure in the J band (2MASS photometric system), we compiled a list of 41 316 676 stars belonging to the PPMXL, UCAC4, and XPM catalogs (Fedorov et al. 2009). The distribution of stars in magnitudes is shown in Fig. 1.

After averaging the differences between the stellar positions and proper motions over the pixels, we formed the differences PPMXL–UCAC4 between the stellar positions and proper motions in the equatorial coordinate system referred to the centers of our pixels. These data were generated for the stars belonging to 12 J magnitude bins with a width of $0\overset{m}{.}5$ with mean values from $10\overset{m}{.}25$ to $15\overset{m}{.}75$. On average, there were up to 1000 stars for each pixel; therefore, the noise level in the averaged proper motion differences decreased approximately by a factor of 30 compared to that in the individual proper motions. From this standpoint, the catalogs of mean differences may be considered as the tables of systematic differences using which the systematic differences for any point on the sphere and any magnitude can be obtained by interpolation. Note that such a representation of systematic differences was used by Mignard and Froeschle (2000) and Farnocchia et al. (2015).

In the second step, the tabular differences were approximated by VSHs in accordance with Eq. (41). The coefficients $t_{nkp}(m)$ and $s_{nkp}(m)$ in this formula were obtained for each mean magnitude of the samples of stars used.

The algorithm for the VSH decomposition of the systematic differences consists of the following steps:

—Determining the indices of the statistically significant harmonics. This procedure consisted in checking conditions (31) at $X = 6.7$, which corresponds to the detection of harmonics with a probability of 0.99 according to the chi-square test. For our pixelization scheme, the limiting value of the index k is $k = 39$, while the highest values of the indices n were determined from conditions (29) and (30).

—Determining the numerical values of the coefficients $t_{nkp}(m_i)$ and $s_{nkp}(m_i)$ and their rms errors $\sigma_s(m_i)$ and $\sigma_t(m_i)$ by the LSM from the set of statistically significant harmonics selected in step 1. The final set of statistically significant harmonics is established using the (2–3) σ criterion. Obviously, the significance level of such a list is 97.7–99.9%.

In the third step, the coefficients $t_{nkp}(m)$ and $s_{nkp}(m)$ were approximated by normalized Legendre polynomials,

$$t_{nkp}(m) = \sum_r t_{nkpr} L_r(\bar{m}), \tag{44}$$
$$s_{nkp}(m) = \sum_r s_{nkpr} L_r(\bar{m}),$$

where

$$L_r(\bar{m}) = \sqrt{\frac{2r+1}{2}} P_r(\bar{m}), \tag{45}$$

and $P_r(\bar{m})$ are Legendre polynomials; the following recurrence relation can be used to calculate the latter:

$$P_{r+1}(\bar{m}) = \frac{2r+1}{r+1}\bar{m}P_r(\bar{m}) - \frac{r}{r+1}P_{r-1}(\bar{m}), \tag{46}$$
$$r = 1, 2, \ldots, \quad P_0 = 1, \quad P_1 = \bar{m}.$$

Table 1. Toroidal decomposition coefficients t_{nkpr} of the field of stellar position differences PPMXL–UCAC4 $\Delta\alpha\cos\delta\,\mathbf{e}_\alpha + \Delta\delta\,\mathbf{e}_\delta$ with the index r due to the magnitude equation. The units of measurement are mas

t_{nkpr}	Value	t_{nkpr}	Value	t_{nkpr}	Value
$t_{1,0,1,0}$	-14.18 ± 0.31	$t_{4,0,1,0}$	6.05 ± 0.31	$t_{7,0,1,1}$	-1.09 ± 0.36
$t_{1,0,1,1}$	-9.51 ± 0.36	$t_{4,0,1,1}$	-2.46 ± 0.36	$t_{7,0,1,2}$	0.90 ± 0.38
$t_{1,0,1,2}$	-2.41 ± 0.38	$t_{4,0,1,2}$	-1.61 ± 0.38	$t_{7,1,1,0}$	-1.92 ± 0.34
$t_{1,1,0,0}$	-2.55 ± 0.29	$t_{4,2,1,0}$	-1.24 ± 0.29	$t_{8,0,1,0}$	1.21 ± 0.31
$t_{1,1,1,0}$	7.89 ± 0.31	$t_{4,2,1,1}$	1.61 ± 0.36	$t_{8,0,1,1}$	-1.10 ± 0.36
$t_{1,1,1,2}$	-1.08 ± 0.38	$t_{5,0,1,0}$	-6.04 ± 0.31	$t_{8,0,1,2}$	-1.54 ± 0.38
$t_{2,0,1,0}$	2.33 ± 0.31	$t_{5,0,1,1}$	-1.60 ± 0.36	$t_{9,1,1,0}$	-3.23 ± 0.32
$t_{2,0,1,1}$	-10.12 ± 0.43	$t_{5,0,1,2}$	-0.80 ± 0.38	$t_{9,1,1,1}$	-2.63 ± 0.39
$t_{2,0,1,2}$	-6.78 ± 0.38	$t_{5,1,0,0}$	2.13 ± 0.29	$t_{10,0,1,0}$	2.65 ± 0.29
$t_{2,0,1,3}$	-4.46 ± 0.40	$t_{5,1,0,1}$	1.50 ± 0.36	$t_{16,0,1,0}$	1.69 ± 0.29
$t_{2,2,0,0}$	-1.84 ± 0.31	$t_{5,4,0,0}$	-1.00 ± 0.29	$t_{16,3,0,0}$	1.96 ± 0.29
$t_{2,2,0,2}$	-3.05 ± 0.38	$t_{6,0,1,0}$	-6.81 ± 0.29	$t_{56,39,1,0}$	2.38 ± 0.29
$t_{3,3,1,0}$	-2.58 ± 0.31	$t_{6,0,1,1}$	-2.51 ± 0.36	$t_{56,39,1,1}$	1.86 ± 0.36
$t_{3,3,1,2}$	1.21 ± 0.38	$t_{7,0,1,0}$	-1.64 ± 0.31		

If $m_{\min} \le m \le m_{\max}$, then the argument of the Legendre polynomials belonging to the interval $[-1; +1]$ is calculated from the formula

$$\bar{m} = 2\frac{m - m_{\min}}{m_{\max} - m_{\min}} - 1. \qquad (47)$$

We established the statistically significant harmonics in index r by taking into account the fact that the same toroidal or spheroidal coefficient with a set of indices nkp could be significant according to the chi-square test for one J sample and insignificant for another. For this reason, the magnitude equation was determined only for those coefficients that turned out to be significant at least in three magnitude samples. In this case, the values for such a coefficient were determined for all twelve J samples. Otherwise, the harmonic was rejected. The coefficients were obtained by solving the systems of equations (44) by the LSM, with the degree of the approximating polynomial having been taken to be three to avoid strong correlations of the sought-for coefficients at our comparatively small number of J samples. In addition, in order that the rms errors of the sought-for coefficients reflect the accuracy of the initial coefficients $t_{nkp}(m)$ and $s_{nkp}(m)$ rather than the accuracy of the formal approximation of the curves $s = s_{nkp}(m)$ and $t = t_{nkp}(m)$, the rms errors of the approximation coefficients $s_{nkpr} = s_{jr}$ and $t_{nkpr} = t_{jr}$ were calculated from the formulas

$$\sigma(s_{jr}) = \sqrt{\sum_{q=0}^{3} w_{rq}^2 \sum_{i=0}^{11} Q_r^2(m_i)\sigma_s^2(m_i)}, \qquad (48)$$

$$\sigma(t_{jr}) = \sqrt{\sum_{q=0}^{3} w_{rq}^2 \sum_{i=0}^{11} Q_r^2(m_i)\sigma_t^2(m_i)},$$

where w_{rq} denote the elements of the inverse matrix of the normal system of equations corresponding to the LSM solution of Eqs. (44), while $\sigma_s(m_i)$ and $\sigma_t(m_i)$ denote the rms errors of the coefficients $t_{nkp}(m_i)$ and $s_{nkp}(m_i)$ found in step 2.

The final toroidal and spheroidal decomposition coefficients t_{nkpr} and s_{nkpr} of the systematic differences between the PPMXL and UCAC4 stellar positions and proper motions in the equatorial coordinate system are given in Tables 1–4.

Table 2. Spheroidal decomposition coefficients s_{nkpr} of the field of stellar position differences PPMXL−UCAC4 $\Delta\alpha\cos\delta\,\mathbf{e}_\alpha + \Delta\delta\,\mathbf{e}_\delta$ with the index r due to the magnitude equation. The units of measurement are mas

s_{nkpr}	Value	s_{nkpr}	Value	s_{nkpr}	Value
$s_{1,0,1,0}$	4.53 ± 0.31	$s_{3,1,1,1}$	1.15 ± 0.36	$s_{6,2,1,1}$	1.91 ± 0.36
$s_{1,0,1,1}$	6.57 ± 0.43	$s_{3,1,1,2}$	-2.23 ± 0.38	$s_{6,4,1,0}$	2.45 ± 0.29
$s_{1,0,1,2}$	4.33 ± 0.38	$s_{3,2,1,0}$	2.41 ± 0.31	$s_{6,5,1,0}$	-2.64 ± 0.29
$s_{1,0,1,3}$	4.65 ± 0.40	$s_{3,2,1,1}$	1.03 ± 0.36	$s_{6,6,1,0}$	1.72 ± 0.29
$s_{1,1,0,0}$	5.81 ± 0.31	$s_{3,2,1,2}$	2.46 ± 0.38	$s_{6,6,1,3}$	-0.91 ± 0.35
$s_{1,1,0,1}$	2.76 ± 0.36	$s_{3,3,1,0}$	-1.91 ± 0.31	$s_{7,4,1,0}$	-2.06 ± 0.29
$s_{1,1,0,2}$	3.00 ± 0.38	$s_{3,3,1,2}$	-1.16 ± 0.38	$s_{8,2,1,0}$	-2.03 ± 0.29
$s_{2,0,1,0}$	6.00 ± 0.31	$s_{4,0,1,1}$	1.30 ± 0.36	$s_{8,2,1,1}$	-1.77 ± 0.36
$s_{2,0,1,1}$	5.86 ± 0.36	$s_{4,0,1,2}$	2.61 ± 0.36	$s_{8,6,1,0}$	-2.54 ± 0.29
$s_{2,0,1,2}$	-1.45 ± 0.38	$s_{4,2,1,0}$	-3.12 ± 0.29	$s_{9,0,1,0}$	2.58 ± 0.31
$s_{2,1,0,0}$	-2.26 ± 0.31	$s_{4,3,1,0}$	2.40 ± 0.31	$s_{9,0,1,1}$	1.05 ± 0.36
$s_{2,1,0,2}$	1.03 ± 0.38	$s_{4,3,1,2}$	1.02 ± 0.38	$s_{9,0,1,2}$	-1.06 ± 0.38
$s_{2,2,0,0}$	-3.56 ± 0.31	$s_{4,4,1,0}$	-2.65 ± 0.29	$s_{10,0,1,0}$	1.99 ± 0.29
$s_{2,2,0,1}$	-1.25 ± 0.36	$s_{5,0,1,0}$	1.43 ± 0.29	$s_{10,0,1,1}$	3.96 ± 0.36
$s_{2,2,0,2}$	-0.90 ± 0.38	$s_{5,0,1,1}$	1.56 ± 0.36	$s_{11,0,1,0}$	2.23 ± 0.29
$s_{2,2,1,0}$	2.91 ± 0.31	$s_{5,4,0,0}$	-1.88 ± 0.29	$s_{11,0,1,1}$	-1.03 ± 0.36
$s_{2,2,1,2}$	-1.02 ± 0.38	$s_{5,4,0,1}$	1.03 ± 0.36	$s_{32,30,0,0}$	2.02 ± 0.29
$s_{3,0,1,0}$	7.29 ± 0.31	$s_{6,0,1,0}$	1.56 ± 0.29	$s_{32,30,0,1}$	1.50 ± 0.36
$s_{3,0,1,1}$	-1.98 ± 0.43	$s_{6,0,1,1}$	3.71 ± 0.36	$s_{51,39,0,0}$	-1.90 ± 0.29
$s_{3,0,1,2}$	-1.25 ± 0.38	$s_{6,1,0,0}$	-2.26 ± 0.31	$s_{51,39,0,1}$	-2.11 ± 0.36
$s_{3,0,1,3}$	1.44 ± 0.40	$s_{6,1,0,1}$	-1.96 ± 0.38		
$s_{3,1,1,0}$	1.88 ± 0.31	$s_{6,2,1,0}$	0.70 ± 0.29		

ANALYSIS OF THE SYSTEMATIC DIFFERENCES

The main purpose of the systematic differences between the positions and proper motions of stars is the possibility to reduce the stellar positions and proper motions from the system of one catalog to the system of another catalog. In our case, this problem is solved for the PPMXL and UCAC4 catalogs using the formula

$$\Delta\mathbf{F}(\alpha,\delta,m) = \sum_{nkpr} t_{nkpr}\mathbf{T}_{nkp}(\alpha,\delta)L_r(\bar{m}) + \sum_{nkpr} s_{nkpr}\mathbf{S}_{nkp}(\alpha,\delta)L_r(\bar{m}). \tag{49}$$

When using this formula to reduce the stellar positions and proper motions from the UCAC4 system

Table 3. Toroidal decomposition coefficients t_{nkpr} of the stellar proper motion differences PPMXL−UCAC4 $\Delta\mu_\alpha \cos\delta\, \mathbf{e}_\alpha + \Delta\mu_\delta\, \mathbf{e}_\delta$ with the index r due to the magnitude equation. The units of measurement are mas and mas yr^{-1}

t_{nkpr}	Value	t_{nkpr}	Value	t_{nkpr}	Value
$t_{1,0,1,0}$	-1.62 ± 0.05	$t_{2,2,1,1}$	0.28 ± 0.04	$t_{7,0,1,2}$	0.12 ± 0.04
$t_{1,0,1,1}$	-0.27 ± 0.05	$t_{3,0,1,0}$	-1.15 ± 0.05	$t_{7,1,1,0}$	0.55 ± 0.05
$t_{1,0,1,2}$	0.72 ± 0.04	$t_{3,0,1,1}$	-0.87 ± 0.04	$t_{7,1,1,2}$	-0.12 ± 0.04
$t_{1,0,1,3}$	0.09 ± 0.04	$t_{3,0,1,2}$	-0.10 ± 0.04	$t_{7,4,1,0}$	0.50 ± 0.04
$t_{1,1,0,0}$	0.74 ± 0.04	$t_{3,1,0,0}$	0.95 ± 0.05	$t_{7,4,1,1}$	0.16 ± 0.04
$t_{1,1,1,0}$	0.89 ± 0.05	$t_{3,1,0,1}$	0.18 ± 0.04	$t_{8,0,1,0}$	0.45 ± 0.04
$t_{1,1,1,1}$	-0.28 ± 0.04	$t_{3,1,0,2}$	-0.10 ± 0.04	$t_{10,0,1,0}$	-0.46 ± 0.04
$t_{1,1,1,2}$	0.16 ± 0.04	$t_{3,3,1,0}$	-0.61 ± 0.05	$t_{10,0,1,1}$	0.15 ± 0.04
$t_{2,0,1,0}$	-3.24 ± 0.05	$t_{3,3,1,2}$	0.09 ± 0.04	$t_{13,0,1,0}$	-0.63 ± 0.05
$t_{2,0,1,1}$	-0.77 ± 0.05	$t_{4,0,1,0}$	0.55 ± 0.04	$t_{13,0,1,2}$	0.10 ± 0.04
$t_{2,0,1,2}$	0.61 ± 0.04	$t_{4,0,1,1}$	0.74 ± 0.05	$t_{13,3,1,0}$	-0.43 ± 0.05
$t_{2,0,1,3}$	-0.31 ± 0.04	$t_{4,0,1,3}$	-0.17 ± 0.04	$t_{13,3,1,1}$	-0.11 ± 0.05
$t_{2,1,0,0}$	-0.31 ± 0.04	$t_{4,1,0,0}$	-0.60 ± 0.05	$t_{14,0,1,0}$	-0.55 ± 0.04
$t_{2,1,0,1}$	0.12 ± 0.04	$t_{5,0,1,0}$	-0.75 ± 0.05	$t_{14,0,1,1}$	-0.23 ± 0.04
$t_{2,1,1,0}$	-0.33 ± 0.04	$t_{5,0,1,1}$	-0.38 ± 0.04	$t_{15,0,1,0}$	0.46 ± 0.04
$t_{2,1,1,1}$	-0.30 ± 0.05	$t_{5,0,1,2}$	0.11 ± 0.04	$t_{15,0,1,1}$	-0.11 ± 0.04
$t_{2,1,1,3}$	0.09 ± 0.04	$t_{5,5,0,0}$	0.54 ± 0.04	$t_{18,0,1,0}$	-0.57 ± 0.04
$t_{2,2,0,0}$	-1.01 ± 0.05	$t_{6,0,1,0}$	-0.63 ± 0.05	$t_{23,0,1,0}$	-0.47 ± 0.04
$t_{2,2,0,1}$	-0.31 ± 0.04	$t_{6,0,1,1}$	-0.47 ± 0.04	$t_{23,0,1,1}$	-0.12 ± 0.04
$t_{2,2,0,2}$	-0.14 ± 0.04	$t_{6,0,1,2}$	0.12 ± 0.04		
$t_{2,2,1,0}$	-0.67 ± 0.04	$t_{7,0,1,0}$	0.47 ± 0.05		

Table 4. Spheroidal decomposition coefficients s_{nkpr} of the stellar proper motion differences PPMXL−UCAC4 $\Delta\mu_\alpha \cos\delta\, \mathbf{e}_\alpha + \Delta\mu_\delta\, \mathbf{e}_\delta$ with the index r due to the magnitude equation. The units of measurement are mas and mas yr^{-1}

s_{nkpr}	Value	s_{nkpr}	Value	s_{nkpr}	Value
$s_{1,0,1,0}$	-3.85 ± 0.05	$s_{3,1,0,0}$	-0.13 ± 0.04	$s_{6,2,1,2}$	-0.10 ± 0.04
$s_{1,0,1,1}$	-1.10 ± 0.05	$s_{3,1,0,1}$	0.25 ± 0.04	$s_{6,5,1,0}$	-0.44 ± 0.04
$s_{1,0,1,2}$	-0.52 ± 0.04	$s_{3,1,1,0}$	0.60 ± 0.05	$s_{6,6,1,0}$	0.66 ± 0.05
$s_{1,0,1,3}$	0.21 ± 0.04	$s_{3,1,1,1}$	-0.18 ± 0.04	$s_{6,6,1,1}$	0.14 ± 0.04
$s_{1,1,0,0}$	3.37 ± 0.04	$s_{3,1,1,2}$	-0.10 ± 0.04	$s_{6,6,1,2}$	-0.10 ± 0.04
$s_{1,1,0,1}$	0.13 ± 0.05	$s_{3,3,0,0}$	-0.35 ± 0.04	$s_{7,0,1,0}$	-0.23 ± 0.04
$s_{1,1,0,3}$	0.18 ± 0.04	$s_{3,3,0,1}$	-0.27 ± 0.04	$s_{7,0,1,1}$	-0.27 ± 0.04
$s_{2,0,1,0}$	-1.67 ± 0.05	$s_{4,0,1,0}$	0.69 ± 0.05	$s_{7,3,0,0}$	-0.41 ± 0.05
$s_{2,0,1,1}$	-0.46 ± 0.05	$s_{4,0,1,1}$	0.42 ± 0.04	$s_{7,3,0,1}$	-0.18 ± 0.05
$s_{2,0,1,2}$	0.20 ± 0.04	$s_{4,0,1,2}$	-0.18 ± 0.04	$s_{7,4,1,0}$	-0.63 ± 0.04
$s_{2,0,1,3}$	-0.13 ± 0.04	$s_{4,2,1,0}$	-0.58 ± 0.05	$s_{8,0,1,0}$	-0.45 ± 0.05
$s_{2,1,0,0}$	1.16 ± 0.05	$s_{4,2,1,1}$	-0.15 ± 0.04	$s_{8,0,1,1}$	0.11 ± 0.04
$s_{2,1,0,1}$	0.49 ± 0.04	$s_{4,2,1,2}$	0.21 ± 0.04	$s_{8,0,1,2}$	0.13 ± 0.04
$s_{2,1,0,2}$	-0.19 ± 0.04	$s_{4,3,0,0}$	-0.59 ± 0.04	$s_{8,2,1,0}$	-0.53 ± 0.05
$s_{2,1,1,0}$	-0.77 ± 0.04	$s_{4,3,1,0}$	0.76 ± 0.04	$s_{8,2,1,2}$	0.13 ± 0.04
$s_{2,1,1,1}$	-0.24 ± 0.04	$s_{4,3,1,1}$	0.18 ± 0.04	$s_{8,3,1,0}$	-0.44 ± 0.04
$s_{2,2,0,0}$	-1.14 ± 0.05	$s_{5,2,0,0}$	-0.58 ± 0.04	$s_{9,5,1,0}$	-0.46 ± 0.04
$s_{2,2,0,2}$	0.13 ± 0.04	$s_{5,2,0,1}$	-0.11 ± 0.04	$s_{10,5,1,0}$	0.48 ± 0.04
$s_{3,0,1,0}$	0.78 ± 0.05	$s_{6,0,1,0}$	-1.06 ± 0.05	$s_{13,5,0,0}$	-0.51 ± 0.04
$s_{3,0,1,1}$	-0.14 ± 0.05	$s_{6,0,1,1}$	0.25 ± 0.04	$s_{13,5,0,1}$	-0.13 ± 0.04
$s_{3,0,1,2}$	-0.16 ± 0.04	$s_{6,0,1,2}$	0.13 ± 0.04	$s_{16,0,1,0}$	-0.69 ± 0.04
$s_{3,0,1,3}$	0.25 ± 0.04	$s_{6,2,1,0}$	0.73 ± 0.05		

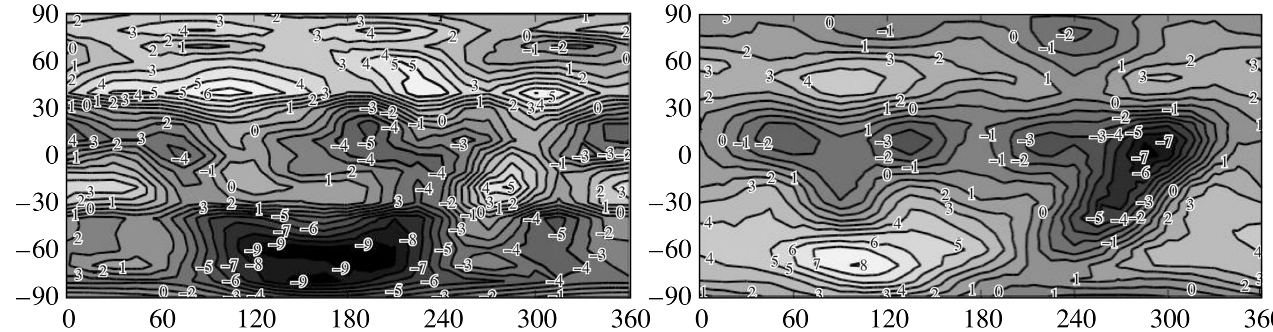

Fig. 2. Maps of systematic position differences PPMXL−UCAC4 for $J = 11^m$. The left and right panels show $\Delta\alpha\cos\delta$ and $\Delta\delta$, respectively. The right ascension (deg) and declination (deg) are along the horizontal and vertical axes, respectively. The units are mas.

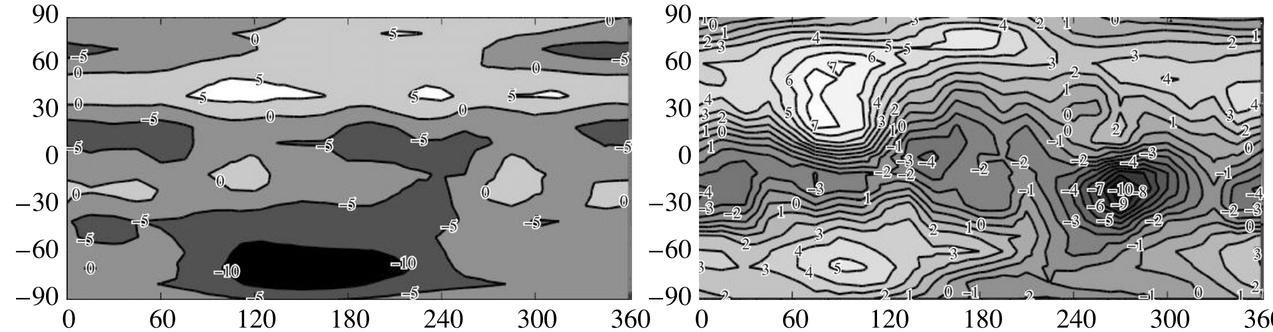

Fig. 3. Maps of systematic position differences PPMXL−UCAC4 for $J = 13^m$. The left and right panels show $\Delta\alpha\cos\delta$ and $\Delta\delta$, respectively. The right ascension (deg) and declination (deg) are along the horizontal and vertical axes, respectively. The units are mas.

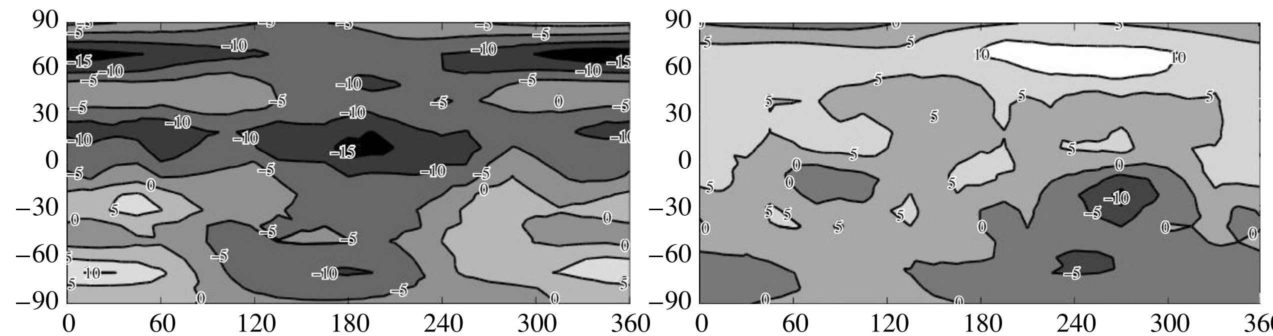

Fig. 4. Maps of systematic position differences PPMXL−UCAC4 for $J = 15^m$. The left and right panels show $\Delta\alpha\cos\delta$ and $\Delta\delta$, respectively. The right ascension (deg) and declination (deg) are along the horizontal and vertical axes, respectively. The units are mas.

to the PPMXL system, the corrections should be calculated with the coefficients t_{nkpr} and s_{nkpr} from Tables 1, 2 and 3, 4, respectively.

Figures 2–4 give an idea of the form of the systematic differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ in the equatorial coordinate system for three magnitudes of stars, $J = 11^m, 13^m, 15^m$, while Figs. 5–7 give the form of the systematic differences $\Delta\mu_\alpha\cos\delta$ and $\Delta\mu_\delta$. In addition, the largest and smallest systematic differences between the positions and proper motions as well as their rms deviations from the means are shown in Tables 5 and 6.

One may think that the decomposition of the systematic differences between the positions and proper motions into systems of orthogonal functions is used only to perform the reduction procedures. However, the systematic differences between the positions and proper motions of the same stars reveal the differences between the reference frames that are realized by the catalogs under consideration. Froeschle and Kovalevsky (1982) showed

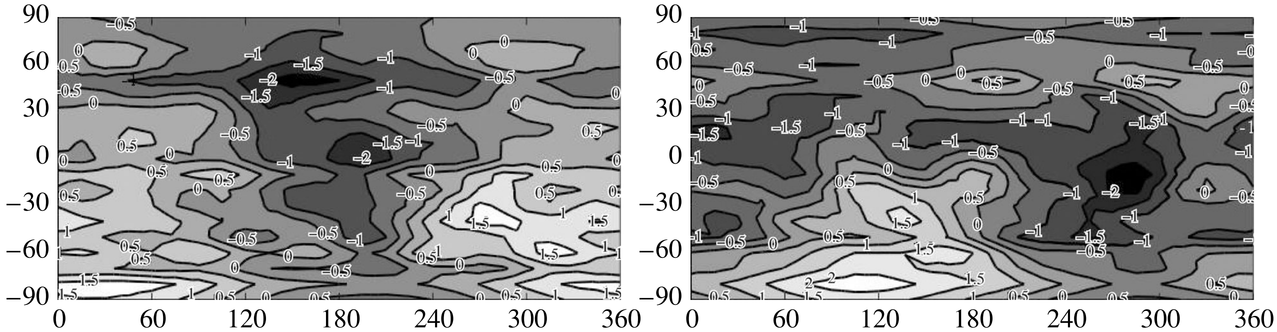

Fig. 5. Maps of systematic proper motion differences PPMXL−UCAC4 for $J = 11^m$. The left and right panels show $\Delta\mu_\alpha \cos\delta$ and $\Delta\mu_\delta$, respectively. The right ascension (deg) and declination (deg) are along the horizontal and vertical axes, respectively. The units of measurement are mas yr^{-1}.

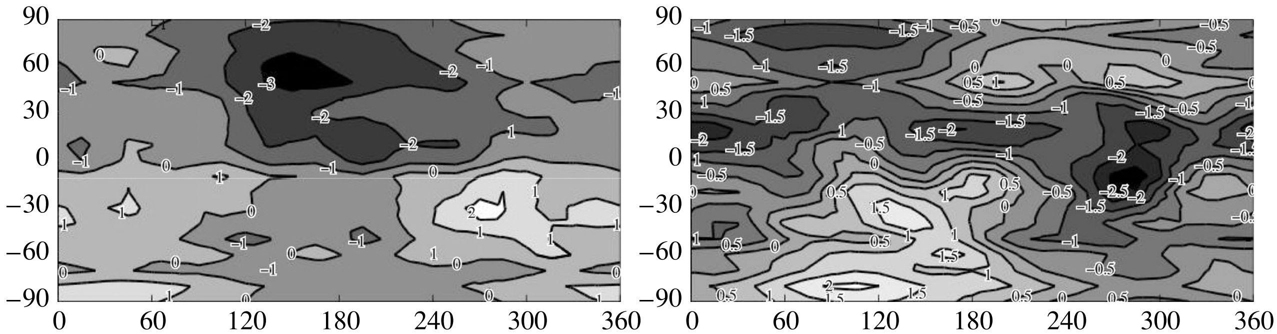

Fig. 6. Maps of systematic proper motion differences PPMXL−UCAC4 for $J = 13^m$. The left and right panels show $\Delta\mu_\alpha \cos\delta$ and $\Delta\mu_\delta$, respectively. The right ascension (deg) and declination (deg) are along the horizontal and vertical axes, respectively. The units of measurement are mas yr^{-1}.

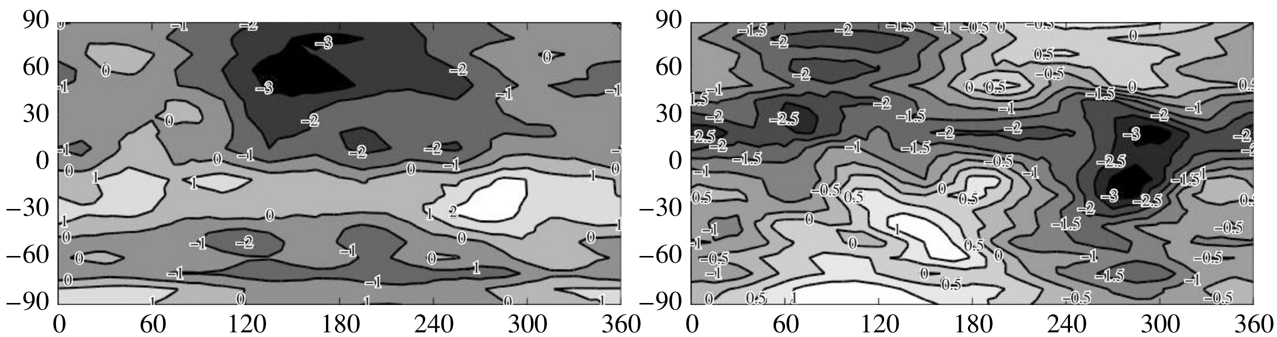

Fig. 7. Maps of systematic proper motion differences PPMXL−UCAC4 for $J = 15^m$. The left and right panels show $\Delta\mu_\alpha \cos\delta$ and $\Delta\mu_\delta$, respectively. The right ascension (deg) and declination (deg) are along the horizontal and vertical axes, respectively. The units of measurement are mas yr^{-1}.

that the rotation angles of the coordinate systems and the rates of their change could be determined by analyzing the systematic differences between the positions and proper motions. The same effects also manifest themselves in the coefficients of the decomposition of the systematic stellar position and proper motion differences into orthogonal functions. Within the model of solid-body rotation, the relationship between the rotation angles of one coordinate system relative to another and the coefficients of the decomposition of the systematic differences between the right ascensions and declinations of stars into scalar harmonics was established by Vityazev and Tsvetkov (1989) and Vityazev (1993). When using VSHs, such a relationship was found by Mignard and Morando (1990). Here, it was shown that the first-order toroidal coefficients in the decomposition of the systematic position differences define the mutual orientation of the reference frames associated with the catalogs under study, while the same coefficients

Table 5. Boundaries of variations and rms deviations from the means for the systematic stellar position differences PPMXL−UCAC4 as a function of the J magnitude of stars, mas

J	$(\Delta\alpha\cos\delta)_{\min}$	$(\Delta\alpha\cos\delta)_{\max}$	$\sigma_{\Delta\alpha\cos\delta}$	$(\Delta\delta)_{\min}$	$(\Delta\delta)_{\max}$	$\sigma_{\Delta\delta}$
11^m	−9.7	6.5	3.6	−7.5	8.1	2.6
13^m	−12.9	7.2	4.1	−10.3	7.8	2.9
15^m	−18.9	10.2	5.3	−11.6	12.0	4.2

Table 6. Boundaries of variations and rms deviations from the means for the systematic stellar proper motion differences PPMXL−UCAC4 as a function of the J magnitude of stars, mas yr^{-1}

J	$(\Delta\mu_\alpha\cos\delta)_{\min}$	$(\Delta\mu_\alpha\cos\delta)_{\max}$	$\sigma_{\Delta\mu_\alpha\cos\delta}$	$(\Delta\mu_\delta)_{\min}$	$(\Delta\mu_\delta)_{\max}$	$\sigma_{\Delta\mu_\delta}$
11^m	−2.23	1.76	0.82	−2.41	2.34	0.87
13^m	−3.50	2.36	1.11	−2.86	2.13	0.96
15^m	−3.84	2.93	1.22	−3.45	1.43	1.02

in the decomposition of the systematic stellar proper motion differences allow the rate of mutual rotation of these frames to be calculated. In the notation of this paper, the working formulas establishing the relationships between the rotation vector components and first-order toroidal coefficients are presented, for example, in Vityazev and Tsvetkov (2009, 2014). As follows from these papers, the mutual orientation angles ϵ_x, ϵ_y, and ϵ_z of the axes of the coordinate systems realized by our catalogs are defined via the VSH decomposition coefficients t_{101}, t_{110}, and t_{111} of the stellar position differences (Table 1). Similarly, the components of the angular velocity of mutual rotation ω_x, ω_y, and ω_z of the axes of the coordinate systems are defined via the VSH decomposition coefficients t_{101}, t_{110}, and t_{111} of the stellar proper motion differences (Table 3). The dependence of these components on the magnitude of stars is shown in Fig. 8.

Obviously, the first-order toroidal coefficients corresponding to the systematic position and proper motion differences allow the position of the pole of the mutual rotation axis on the celestial sphere to be determined. Figure 9 shows the mutual rotation angles around the pole, the coordinates of the pole, and a vector map of the components of the systematic position differences that are formed by these rotations for stars with various magnitudes. We see that the rotation angles change for different magnitude groups and can reach ∼10 mas. The analogous data characterizing the rates of mutual rotation are shown in Fig. 10. It follows from these figures that the rate of mutual rotation of the PPMXL and UCAC4 reference frames can reach 0.7 mas yr^{-1}. Since the absolute value of the systematic difference does not exceed the sum of the absolute values of the errors for our catalogs, it can be said that the absolute values of the errors for each catalog are 0.35 mas yr^{-1}. If we recall that the measure of inertiality of the Hipparcos reference frame (the accuracy with which the Hipparcos catalog is tied to the ICRS) is defined by 0.25 mas yr^{-1} (Perryman et al. 1997), then it can be concluded that the residual rotations of the PPMXL and UCAC4 frames essentially reproduce the measure of inertiality of the HCRF.

In conclusion, it should be said that the systematic stellar position and proper motion differences PPMXL−UCAC4 show a pronounced dependence on the magnitude of stars. This manifests itself in the fact that almost all coefficients t_{nkp} and s_{nkp} are functions of the magnitude (Tables 1−4). Accordingly, there are marked differences in the maps of systematic differences (Figs. 2−7) and in the mutual orientation and spin of the PPMXL and UCAC4 reference frames (Figs. 9 and 10). A strong manifestation of the magnitude equation in the mutual orientation and rotation of the coordinate systems around the z axis (Fig. 8) deserves our attention. We see that the PPMXL

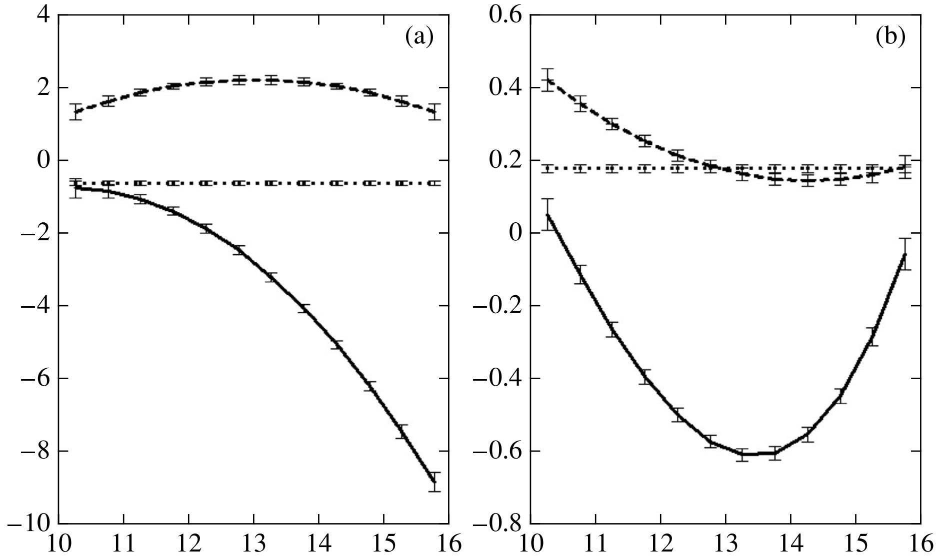

Fig. 8. (a) Mutual orientation angles (mas) of the PPMXL and UCAC4 reference frames: ϵ_x (dashes), ϵ_y (dots), and ϵ_z (solid line). (b) Components of the angular velocity of mutual rotation (mas yr^{-1}) of the PPMXL and UCAC4 reference frames: ω_x (dashes), ω_y (dots), and ω_z (solid line). The J magnitudes of the samples are along the horizontal axes.

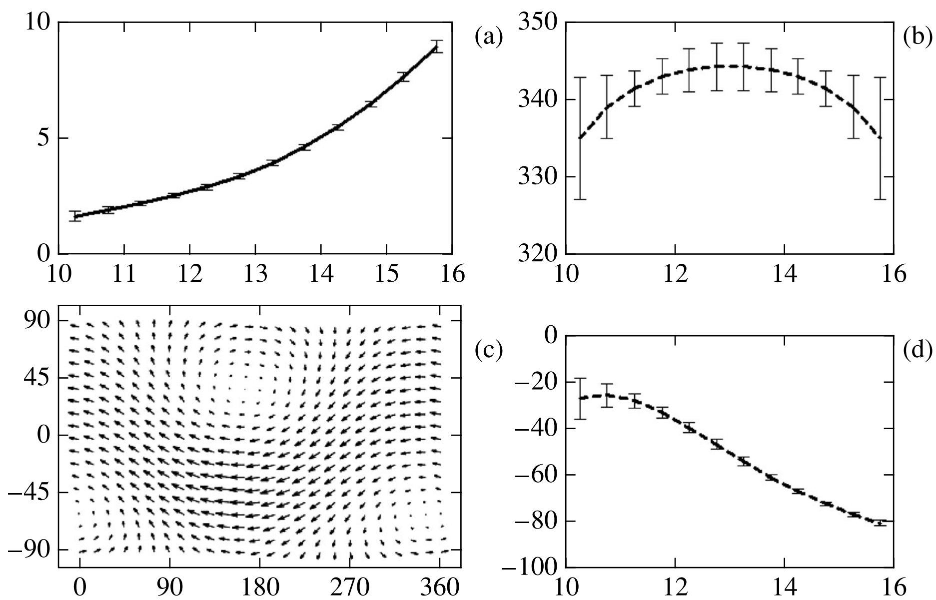

Fig. 9. (a) Mutual rotation angles (mas) of the PPMXL and UCAC4 reference frames around the pole whose right ascensions and declinations in degrees are shown on panels (b, d). The J magnitudes of the samples are along the horizontal axes. (c) The vector field of systematic differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ corresponding to the position of the pole at $J = 13^m$. The right ascensions (deg) and declinations (deg) are along the horizontal and vertical axes, respectively.

zero point of right ascensions relative to the UCAC4 zero point at epoch J2000 decreases monotonically as we pass to faint stars and reaches −10 mas for $J = 16^m$. Because of the pronounced extremum of the component ω_z at $J = 13^m$, this zero point is shifted considerably in the direction of decrease with time under mutual rotation of our reference frames. Table 7 gives a quantitative estimate of the influence of the magnitude equation on the systematic differences. It shows the ranges of systematic differences for various magnitudes. We see that a minimal manifestation of the magnitude equation is observed for the bright stars of our range ($J = 11$), while the range of systematic position differences increases approximately by a factor of 2 when passing from $J = 11$ to 15. The same increase is also observed for the range of

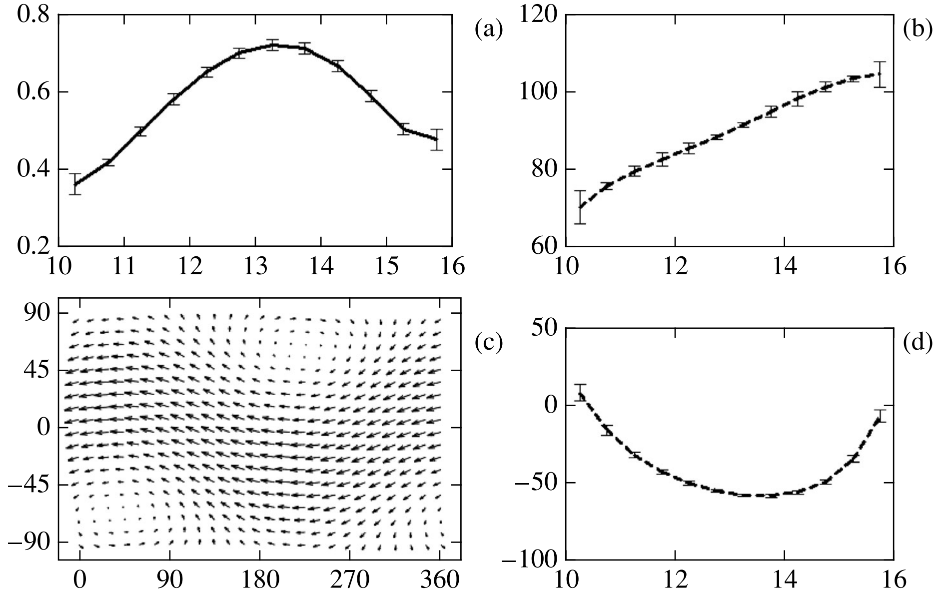

Fig. 10. (a) Angular velocities (mas yr^{-1}) of mutual rotation of the PPMXL and UCAC4 reference frames around the pole whose right ascensions and declinations are shown on panels (b, d). The J magnitudes of the samples are along the horizontal axes. (c) The vector field of systematic differences $\Delta\mu_\alpha \cos\delta$ and $\Delta\mu_\delta$ corresponding to the position of the pole at $J = 13^m$. The right ascensions (deg) and declinations (deg) are along the horizontal and vertical axes, respectively.

systematic differences between the proper motions in right ascension, while the range of systematic differences between the proper motions in declination changes only slightly. In this case, it is important to emphasize that the extreme values of the systematic position and proper motion differences do not exceed in absolute value 20 mas and 4 mas yr^{-1}, respectively (Tables 5 and 6). At the same time, the analogous extreme differences before applying the procedure of approximating the results of our VSH pixelization of the differences are 60 mas and 8 mas yr^{-1} (if the unrealistic outliers, greater than 100 mas in positions and 25 mas yr^{-1} in proper motions in some cases, are disregarded). All of this suggests that the systematic difference between the PPMXL and UCAC4 is less than their random difference.

Table 7. Ranges of systematic stellar position and proper motion differences PPMXL–UCAC4 as a function of the J magnitude of stars

J	ΔRA	ΔDEC	$\Delta PMRA$	$\Delta PMDEC$
11^m	17.2	15.1	3.99	4.75
13^m	20.1	18.1	5.86	4.99
15^m	29.1	23.6	6.77	4.88

$\Delta RA = (\Delta\alpha\cos\delta)_{\max} - (\Delta\alpha\cos\delta)_{\min}$, $\Delta DEC = (\Delta\delta)_{\max} - (\Delta\delta)_{\min}$ (mas), $\Delta PMRA = (\Delta\mu_\alpha\cos\delta)_{\max} - (\Delta\mu_\alpha\cos\delta)_{\min}$, $\Delta PMDEC = (\Delta\mu_\delta)_{\max} - (\Delta\mu_\delta)_{\min}$ (mas yr^{-1}).

ACKNOWLEDGMENTS

This work was supported by grant no. 6.37.343.2015 of the St. Petersburg State University.

REFERENCES

1. G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1966; Atomizdat, Moscow, 1970).
2. R. Bien, W. Fricke, T. Lederle, and H. Schwn, Veroff. Heidelberg Astron. Rech.-Inst. **29**, 1 (1978).
3. P. Brosche, Veroff. Heidelberg Astron. Rech.-Inst. **17**, 1 (1966).
4. D. Farnocchia, S. R. Chesley, A. B. Chamberlin, and D. J. Tholen, Icarus **245**, 94 (2015).
5. P. N. Fedorov, A. A. Myznikov, and V. S. Akhmetov, Mon. Not. R. Astron. Soc. **393**, 133 (2009).
6. M. Froeschle and J. Kovalevsky, Astron. Astrophys. **116**, 89 (1982).
7. K. M. Gorski, E. Hivon, A. J. Banday, B. D. Wandelt, F. K. Hansen, M. Reinecke, and M. Bartelmann, Astrophys. J. **622**, 759 (2005).
8. F. Mignard and S. Klioner, Astron. Astrophys. **547**, A59 (2012).
9. F. Mignard and B. Morando, in *Proceedings of the Journées 1990: Systèmes de référence spatio-temporels, Colloque André Danjon* (1990), p. 151.
10. F. Mignard and M. Froeschle, Astron. Astrophys. **354**, 732 (2000).

11. M. A. C. Perriman et al., *The HIPPARCOS and TYCHO Catalogues,* ESA SP-1200 (ESA, 1997), Vols. 1–17.
12. S. Roeser, M. Demleitner, and E. Schilbach, Astron. J. **139**, 2440 (2010).
13. M. F. Skrutskie, R. M. Cutri, R. Stiening, M. D. Weinberg, S. Schneider, J. M. Carpenter, C. Beichman, R. Capps, et al., Astron. J. **131**, 1163 (2006).
14. V. V. Vityazev, Astron. Astrophys. Trans. **4**, 195 (1993).
15. V. V. Vityazev and A. S. Tsvetkov, Vestn. Leningr. Univ., Ser. 1, No. 2, 73 (1989).
16. V. V. Vityazev and A. S. Tsvetkov, Astron. Lett. **35**, 100 (2009).
17. V. V. Vityazev and A. S. Tsvetkov, Astron. Nachr. **334**, No. 8 (2013).
18. V. V. Vityazev and A. S. Tsvetkov, Mon. Not. R. Astron. Soc. **442**, 1249 (2014).
19. N. Zacharias, C. T. Finch, T. M. Girard, A. Henden, J. L. Bartlett, D. G. Monet, and M. I. Zacharias, Astron. J. **145**, 44 (2013).

Translated by V. Astakhov