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Analysis of the Three-Dimensional Stellar Velocity Field Using Vector Spherical Functions

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Abstract—We apply vector spherical functions to problems of stellar kinematics. Using these functions allows all of the systematic components in the stellar velocity field to be revealed without being attached to a specific physical model. Comparison of the theoretical decomposition coefficients of the equations for a particular kinematic model with observational data can provide precise information about whether the model is compatible with the observations and can reveal systematic components that are not described by this model. The apparatus of vector spherical functions is particularly well suited for analyzing the present and future (e.g., GAIA) catalogs containing all three velocity vector components: the proper motions in both coordinates and the radial velocity. We show that there are systematic components in the proper motions of Hipparcos stars that cannot be interpreted in terms of the linear Ogorodnikov—Milne model. The same result is also confirmed by an analysis of the radial velocities for these stars.

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INTRODUCTION

The appearance of the large-scale Hipparcos, Tycho-2, USAC-2, and USNO stellar catalogs of proper motions provides qualitatively new material for investigating the kinematics of Galactic stars. The prospects of highly accurate parallax, proper motion, and radial velocity measurements for hundreds of millions of stars planned in the GAIA project are an incentive for developing new methods of kinematic analysis of stars.

The study of stellar kinematics is usually based on estimating the parameters of the models that describe the stellar velocity field components by the least-squares method. This approach is methodologically impeccable, provided that the models used are complete. In reality, we can never include all of the phenomena related to stellar kinematics in the model, i.e., make the model complete from a physical viewpoint. There exist quite a few alternative, more or less complex models of the stellar velocity field, but choosing between them is a complex problem. A different approach to constructing models based on the representation of the data under study by complete orthogonal sets of functions (Brosche 1966; Schwan 2001; Vityazev and Tsvetkov 1989; Vityazev 1989, 1993; Tsvetkov 1997; Tsvetkov and Popov 2006) can be suggested for its solution. Such models are complete (from a mathematical viewpoint), i.e., all of the information available in the observational data can be described using the coefficients of their decomposition into the functions of a chosen basis.

Thus, the formal models of the stellar velocity field allow us, first, to determine whether systematic components are present in the proper motions and radial velocities and only then to choose a specific physical model. In addition, comparison of the theoretical decomposition of the equations for a physical model into a chosen set of orthogonal functions with the coefficients derived from a particular sample of stars can provide precise information about whether the adopted model is compatible with the observations. In the cases where the correspondence between the physical and mathematical model parameters can be obtained, the method of orthogonal representations is able to give an estimate of the physical model parameters protected from any distortions (biases) from the phenomena that were not included in the model. This approach was probably first implemented in our papers (Vityazev and Tsvetkov 1989; Vityazhev 1993) using scalar spherical functions.

Therefore, it seems appropriate to use the method of decomposing the stellar velocity field into a set of

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three-dimensional vector spherical functions (VSFs) to study the stellar kinematics. Note that two-dimensional VSFs were first used by Mignard and Morando (1990) in astrometric problems related to the comparison of catalogs to represent the systematic differences between Hipparcos and FK5. Vityazev and Shuksto (2004, 2005) extended this approach to a kinematic analysis of only the stellar proper motions without including the stellar radial velocities. The goal of this paper is to use three-dimensional VSFs to study the total stellar velocity field whose components are determined by the measurements of both stellar proper motions and radial velocities.

SCALAR SPHERICAL FUNCTIONS

Spherical functions are widely used in various areas of mathematics and physics. Their definition can be found in many sources, for example in Arfken (1970). In this paper, we will use the following representation for them:

$$K_{nkp}(l,b) = R_{nk} \times \begin{cases} P_{n,0}(b), & k=0, \quad p=1, \\ P_{nk}(b)\sin kl, & k\neq 0, \quad p=0, \\ P_{nk}(b)\cos kl, & k\neq 0, \quad p=1, \end{cases}$$

$$R_{nk} = \sqrt{\frac{2n+1}{4\pi}} \begin{cases} \sqrt{\frac{2(n-k)!}{(n+k)!}}, & k>0, \\ 1, & k=0, \end{cases}$$

where l and b are the longitude and latitude of the point on the sphere, respectively ($0 \le l \le 2\pi$; $-\pi/2 \le b \le \pi/2$); $P_{nk}(b)$ are the Legendre (at $k=0$) and associated Legendre (for $k>0$) polynomials that can be calculated using the recurrence relations

$$P_{nk}(b) = \sin b \frac{2n-1}{n-k} P_{n-1,k}(b) - \frac{n+k-1}{n-k} P_{n-2,k}(b),$$

$$k = 0, 1, \ldots, n = k+1, k+2, \ldots$$

$$P_{kk}(b) = \frac{(2k)!}{2^k k!} \cos^k b,$$

$$P_{k+1,k}(b) = \frac{(2k+2)!}{2^{k+1}(k+1)!} \cos^k b \sin b.$$

VECTOR SPHERICAL FUNCTIONS

Consider a set of mutually orthogonal unit vectors $\mathbf{e}_l$, $\mathbf{e}_b$, and $\mathbf{e}_r$ in the directions of the change in longitude and latitude and along the line of sight, respectively, in a plane tangential to the sphere. Using the definitions of VSFs in Arfken (1970) or Varshalovich et al. (1975), let us introduce radial, $\mathbf{V}_{nkp}$, toroidal, $\mathbf{T}_{nkp}$, and spheroidal, $\mathbf{S}_{nkp}$, VSFs via the relations

$$\mathbf{V}_{nkp}(l,b) = K_{nkp}(l,b)\mathbf{e}_r,$$

$$\mathbf{T}_{nkp} = \frac{1}{\sqrt{n(n+1)}} \times \left(\frac{\partial K_{nkp}(l,b)}{\partial b}\mathbf{e}_l - \frac{1}{\cos b}\frac{\partial K_{nkp'}(l,b)}{\partial l}\mathbf{e}_b \right),$$

$$\mathbf{S}_{nkp} = \frac{1}{\sqrt{n(n+1)}} \times \left(\frac{1}{\cos b}\frac{\partial K_{nkp}(l,b)}{\partial l}\mathbf{e}_l + \frac{\partial K_{nkp}(l,b)}{\partial b}\mathbf{e}_b \right).$$

Denote the components of the unit vector $\mathbf{e}_l$ by T^l_{nkp} and S^l_{nkp} and the components of the unit vector $\mathbf{e}_b$ by T^b_{nkp} and S^b_{nkp}, respectively:

$$\mathbf{T}_{nkp} = T^l_{nkp}\mathbf{e}_l + T^b_{nkp}\mathbf{e}_b,$$

$$\mathbf{S}_{nkp} = S^l_{nkp}\mathbf{e}_l + S^b_{nkp}\mathbf{e}_b.$$

These components are defined as

$$T^l_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \times \begin{cases} P_{n,1}(b), \quad k=0, \quad p=1, \\ (-k\,\mathrm{tg}\, b P_{nk}(b) + P_{n,k+1}(b))\sin kl, \\ k\neq 0, \quad p=0, \\ (-k\,\mathrm{tg}\, b P_{nk}(b) + P_{n,k+1}(b))\cos kl, \\ k\neq 0, \quad p=1; \end{cases}$$

$$T^b_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \times \begin{cases} 0, & k=0, \quad p=1, \\ -\frac{k}{\cos b}P_{nk}(b)\cos kl, & k\neq 0, \quad p=0, \\ +\frac{k}{\cos b}P_{nk}(b)\sin kl, & k\neq 0, \quad p=1; \end{cases}$$

$$S^l_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}} \times \begin{cases} 0, & k=0, \quad p=1, \\ -\frac{k}{\cos b}P_{nk}(b)\cos kl, & k\neq 0, \quad p=0, \\ -\frac{k}{\cos b}P_{nk}(b)\sin kl, & k\neq 0, \quad p=1; \end{cases}$$

$$S^b_{nkp} = \frac{R_{nk}}{\sqrt{n(n+1)}}$$

$$\times \begin{cases} P_{n,1}(b), \quad k=0, \quad p=1; \\ (-k\,\mathrm{tg}\,bP_{nk}(b)+P_{n,k+1}(b))\sin kl, \\ k\neq 0, \quad p=0, \\ (-k\,\mathrm{tg}\,bP_{nk}(b)+P_{n,k+1}(b))\cos kl, \\ k\neq 0, \quad p=1, \end{cases}$$

For convenience, a linear numeration of the functions $\mathbf{V}_{nkp}$, $\mathbf{T}_{nkp}$, and $\mathbf{S}_{nkp}$ by one index j is often introduced, where

$$j = n^2 + 2k + p - 1.$$

The introduced functions satisfy the relations

$$\iint_\Omega (\mathbf{V}_i \cdot \mathbf{V}_j)\, d\omega = \iint_\Omega (\mathbf{T}_i \cdot \mathbf{T}_j)\, d\omega$$

$$= \iint_\Omega (\mathbf{S}_i \cdot \mathbf{S}_j)\, d\omega = \begin{cases} 0, & i\neq j, \\ 1, & i=j; \end{cases}$$

$$\iint_\Omega (\mathbf{V}_i \cdot \mathbf{T}_j)\, d\omega = \iint_\Omega (\mathbf{V}_i \cdot \mathbf{S}_j)\, d\omega$$

$$= \iint_\Omega (\mathbf{S}_i \cdot \mathbf{T}_j)\, d\omega = 0, \quad \forall i,j.$$

In other words, the set of functions $\mathbf{V}_{nkp}$, $\mathbf{T}_{nkp}$, and $\mathbf{S}_{nkp}$ forms an orthonormal set of functions on the sphere.

REPRESENTATION OF THE STELLAR VELOCITY FIELD BY A SET OF VSFs

Consider the real stellar velocity field on the celestial sphere:

$$\mathbf{U}(l,b) = V_r/r\mathbf{e}_r + \mathcal{K}\mu_l \cos b\mathbf{e}_l + \mathcal{K}\mu_b\mathbf{e}_b,$$

where V_r is the radial velocity, μ_l and μ_b are the stellar proper motion components in Galactic longitude and latitude, r is the distance to the star, and $\mathcal{K} = 4.738$ is the conversion factor of the dimensions of stellar proper motions mas yr^{-1} into km s^{-1} kpc^{-1}.

Using the set of VSFs defined above, we can decompose the velocity field as

$$\mathbf{U}(l,b) = \sum_{nkp} v_{nkp}\mathbf{V}_{nkp} \tag{1}$$

$$+ \sum_{nkp} t_{nkp}\mathbf{T}_{nkp} + \sum_{nkp} s_{nkp}\mathbf{S}_{nkp},$$

where, since the basis is orthonormal, the decomposition coefficients v_{nkp}, t_{nkp}, and s_{nkp} are calculated from the formulas

$$v_{nkp} = \iint_\Omega (\mathbf{U}\cdot\mathbf{V}_{nkp})\, d\omega$$

$$= \int_0^{2\pi} dl \int_{-\pi/2}^{+\pi/2} V_r(l,b)/rV_{nkp} \cos b\, db,$$

$$t_{nkp} = \iint_\Omega (\mathbf{U}\cdot\mathbf{T}_{nkp})\, d\omega$$

$$= \int_0^{2\pi} dl \int_{-\pi/2}^{+\pi/2} \left(\mathcal{K}\mu_l \cos bT^l_{nkp} + \mathcal{K}\mu_b T^b_{nkp}\right) \cos b\, db,$$

$$s_{nkp} = \iint_\Omega (\mathbf{U}\cdot\mathbf{S}_{nkp})\, d\omega$$

$$= \int_0^{2\pi} dl \int_{-\pi/2}^{+\pi/2} \left(\mathcal{K}\mu_l \cos bS^l_{nkp} + \mathcal{K}\mu_b S^b_{nkp}\right) \cos b\, db.$$

DECOMPOSITION OF THE OGORODNIKOV–MILNE EQUATIONS INTO A SET OF VSFs

The equations of the Ogorodnikov–Milne model (du Mont 1977; Rybka 2004) are commonly used in analyzing the stellar proper motions. In this model, the stellar velocity field is represented by a linear expression,

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{\Omega}\times\mathbf{r} + \mathbf{M}^+\times\mathbf{r}, \tag{2}$$

where $\mathbf{V}$ is the stellar velocity, $\mathbf{V}_0$ is the effect of the translational solar motion, $\mathbf{\Omega}$ is the angular velocity of rigid-body rotation of the stellar system, and $\mathbf{M}^+$ is the symmetric velocity field deformation tensor.

The Ogorodnikov–Milne model contains 12 parameters:

U, V, W the components of the velocity vector of translational solar motion $\mathbf{V}_0$ relative to the stars;

ω_1, ω_2, ω_3 are the components of the vector of rigid-body rotation $\mathbf{\Omega}$;

M^+_{11}, M^+_{22}, M^+_{33} are the parameters of the tensor $\mathbf{M}^+$ that describe the velocity field contraction–expansion along the principal Galactic axes;

M^+_{12}, M^+_{13}, M^+_{23} are the parameters of the tensor $\mathbf{M}^+$ that describe the velocity field deformation in the principal plane and in the two planes perpendicular to it.

Projecting Eq. (2) onto the unit vectors of the Galactic coordinate system yields

$$V_r/r = -(U/r)\cos l\cos b - (V/r)\sin l\cos b \tag{3}$$

$$-\,(W/r)\sin b + M^+_{13}\sin 2b\cos l$$

$$+\, M^+_{23}\sin 2b\sin l + M^+_{12}\cos^2 b\sin 2l$$

$$+ M^+_{11}\cos^2 b\cos^2 l + M^+_{22}\cos^2 b\sin^2 l + M^+_{33}\sin^2 b,$$

$$\mathcal{K}\mu_l\cos b = (U/r)\sin l - (V/r)\cos l - \omega_1\sin b\cos l - \omega_2\sin b\sin l + \omega_3\cos b - M^+_{13}\sin b\sin l + M^+_{23}\sin b\cos l + M^+_{12}\cos b\cos 2l - \frac{1}{2}M^+_{11}\cos b\sin 2l + \frac{1}{2}M^+_{22}\cos b\sin 2l, \quad (4)$$

$$\mathcal{K}\mu_b = (U/r)\cos l\sin b + (V/r)\sin l\sin b - (W/r)\cos b + \omega_1\sin l - \omega_2\cos l - \frac{1}{2}M^+_{12}\sin 2b\sin 2l + M^+_{13}\cos 2b\cos l + M^+_{23}\cos 2b\sin l - \frac{1}{2}M^+_{11}\sin 2b\cos^2 l - \frac{1}{2}M^+_{22}\sin 2b\sin^2 l + \frac{1}{2}M^+_{33}\sin 2b. \quad (5)$$

Since there is a linear relation between the coefficients M^+_{11}, M^+_{22}, and M^+_{33} in Eqs. (4) and (5), the substitutions $M^*_{11} = M^+_{11} - M^+_{22}$ and $M^*_{33} = M^+_{33} - M^+_{22}$ are often introduced when the stellar proper motions are analyzed (du Mont 1977).

Let us decompose Eqs. (3)–(5) into VSFs to ascertain which parameters of this model are responsible for specific harmonics in the decomposition of the stellar proper motions. It should be kept in mind that the components of the solar motion enter into Eqs. (3) and (5) with the factor $1/r$. This means that the effects of the solar motion should be eliminated when VSFs are used. Otherwise, the solution is meaningful only for stars at approximately equal heliocentric distances; in this case, we will be able to determine the parameters of the solar motion only to within the factor $1/\langle r\rangle$. The results of our theoretical decomposition of the right-hand sides of Eqs. (3)–(5) are presented in Table 1.

In the simplified form of the Ogorodnikov–Milne model that is usually called the Oort–Lindblad model (Ogorodnikov 1965), the stellar system is assumed to to rotate exactly in the Galactic plane, i.e., ω_1 and ω_2 are zero; the contraction–expansion along the axes is not considered, while the deformation exists only in the Galactic plane and is described by the Oort parameter $A = M_{12}$. The number of decomposition coefficients will decrease accordingly.

The formulas from Table 2 can be used to solve the inverse problem, i.e., to determine the Ogorodnikov–Milne model parameters via the VSF decomposition coefficients of the stellar radial velocities and proper motions.

Table 1. VSF decomposition coefficients of the Ogorodnikov–Milne equations

Coefficient	Value
v_{001}	$1.18M^+_{11} + 1.18M^+_{22} + 1.18M^+_{33}$
v_{101}	$-2.05W/\langle r\rangle$
v_{110}	$-2.05V/\langle r\rangle$
v_{111}	$-2.05U/\langle r\rangle$
v_{201}	$-0.53M^+_{11} - 0.53M^+_{22} + 1.06M^+_{33}$
v_{210}	$1.83M^+_{23}$
v_{211}	$1.83M^+_{13}$
v_{220}	$1.83M^+_{12}$
v_{221}	$0.92M^+_{11} - 0.92M^+_{22}$
t_{101}	$2.89\omega_3$
t_{110}	$2.89\omega_2$
t_{111}	$2.89\omega_1$
s_{101}	$-2.89W/\langle r\rangle$
s_{110}	$-2.89V/\langle r\rangle$
s_{111}	$-2.89U/\langle r\rangle$
s_{201}	$-0.65M^+_{11} - 0.65M^+_{22} + 1.29M^+_{33}$
s_{210}	$2.24M^+_{23}$
s_{211}	$2.24M^+_{13}$
s_{220}	$2.24M^+_{12}$
s_{221}	$1.12M^+_{11} - 1.12M^+_{22}$

Based on the results presented in Table 1, we can formulate the following properties of the decomposition of Eqs. (3)–(5):

— the Ogorodnikov–Milne model is completely described by the set of harmonics whose order in index n does not exceed two;

— the solar components are determined (to within the factor of the mean distance to the stars) via the radial and spheroidal coefficients with indices 101, 110, and 111 independently from the stellar radial velocities and proper motions;

— the velocity field deformation parameters M^+_{23}, M^+_{13}, and M^+_{12} are also determined independently from the stellar radial velocities and proper motions via the radial and spheroidal coefficients with indices 210, 211, and 220;

— the contraction–expansion parameters M^+_{11}, M^+_{22}, and M^+_{33} are completely determined from the stellar

Table 2. Relationships of the Ogorodnikov–Milne model parameters to the VSF decomposition coefficients of the stellar radial velocities and proper motions

Parameter	Formula
$W/\langle r\rangle$	$-0.488v_{101}$
$V/\langle r\rangle$	$-0.488v_{110}$
$U/\langle r\rangle$	$-0.488v_{111}$
M^+_{11}	$0.282v_{001} - 0.314v_{201} + 0.543v_{221}$
M^+_{22}	$0.282v_{001} - 0.314v_{201} - 0.543v_{221}$
M^+_{33}	$0.282v_{001} + 0.629v_{201}$
M^+_{23}	$0.546v_{210}$
M^+_{13}	$0.546v_{211}$
M^+_{12}	$0.546v_{220}$
ω_3	$0.346t_{101}$
ω_2	$0.346t_{110}$
ω_1	$0.346t_{111}$
$W/\langle r\rangle$	$-0.346s_{101}$
$V/\langle r\rangle$	$-0.346s_{110}$
$U/\langle r\rangle$	$-0.346s_{111}$
$M^+_{11} - M^+_{22}$	$0.893s_{221}$
$M^+_{33} - M^+_{22}$	$0.769s_{201} + 0.446s_{221}$
M^+_{23}	$0.446s_{210}$
M^+_{13}	$0.446s_{211}$
M^+_{12}	$0.446s_{220}$

radial velocities via the radial coefficients with indices 001, 201, 221;

– the spheroidal coefficients with indices 201 and 221 allow the contraction–expansion parameters to be determined to within the constant M^+_{22}, which is usually set equal to zero (Clube 1972);

– the rigid-body rotation velocity components for the sample of stars being analyzed ω_1, ω_2, and ω_3 are determined only from the toroidal coefficients with indices 111, 110, and 101;

– In the VSF method, with the exception of the rigid-body rotation parameters, the solution from the radial functions completely solves the problem of determining the velocity field parameters only from the radial velocities independently from the stellar proper motions. This property allows the compatibility of the stellar radial velocities and proper motions relative to the chosen kinematic model to be tested. Such a test is highly desirable, since the stellar radial velocities and proper motions are determined by fundamentally different methods and can have their own systematic errors.

THE VSF METHOD IN PRACTICE

We will assume that we have a catalog of stars at our disposal with known parallaxes, coordinates, radial velocities, and proper motion components in Galactic longitude and latitude. Before describing the main steps of the VSF method, let us consider one fundamental point.

In principle, the sought-for decomposition coefficients can be determined by the standard least-squares method. In accordance with the specificity of the VSF method, we should obtain a separate solution from the first component of Eq. (1) (radial velocities) and a joint solution from the two remaining components of this equation (stellar proper motions).

However, for a nonuniform distribution of stars over the sphere, we run into two difficulties. First, cross correlations appear between the unknowns, which, in the long run, reduce the reliability of their numerical estimates. Second, "ghosts", i.e., harmonics that have no particular physical nature but are caused exclusively by the distribution of stars over the sphere, appear.

A detailed analysis of this situation and numerical experiments showed that a preaveraging of the data over certain fields on the celestial sphere is apparently the simplest and fairly efficient method of combatting these shortcomings. This technique, known as the problem of data pixelization on a sphere, is now widely used to analyze the CMB anisotropy (Bennett 1996, 2003). As applied to our problem, the pixelization scheme should meet the requirement that the pixel centers follow uniformly in both latitude and longitude. Two schemes meet this requirement. One of these is known as HEALPix (Gorski 2005). In this method, the areas of all spherical fields are equal and their centers are located on equidistant (in latitude) parallels. The centers on each parallel are equally spaced in longitude, with the longitude step being the same in the equatorial zone and changing from parallel to parallel in the polar zones.

The second pixelization method is the so-called equidistant cylindrical projection (ECP), in which the celestial sphere is broken down into spherical trapeziums by dividing the equator and the meridian semicircle into M and N parts, respectively. The indices m and n, which describe the field number in longitude and the zone number in latitude, can be easily calculated from the stellar coordinates using the formulas

$$m = \left[\frac{l}{360^\circ}M\right] + 1; \quad n = \left[\frac{90^\circ - b}{180^\circ}N\right] + 1,$$

where $[x]$ is the smallest integer that does not exceed x.

The coordinates of the field center are defined as

$$l_m = \frac{180^\circ}{M} + \frac{360^\circ}{M}(m-1), \quad m = 1, 2, \ldots, M,$$

$$b_n = 90^\circ - \frac{90^\circ}{N} - \frac{180^\circ}{N}(n-1), \quad n = 1, 2, \ldots, N.$$

In both cases, for each field with number $P = 0, 1, \ldots, N_{\rm pix}$, the average values of the proper motions $\mu_l^*(P) = \langle \mathcal{K}\mu_l \cos b\rangle$ and $\mu_b^*(P) = \langle \mathcal{K}\mu_b\rangle$ and radial velocities $V^*(P) = \langle V_r/r\rangle$ of the stars falling into this field are calculated. After these averagings, the sought-for coefficients can be calculated from the formulas

$$v_{nkp} = \sum_{P=0}^{N_{pix}-1} V^*(P)V_{nkp}(P)W_P, \qquad (6)$$

$$t_{nkp} = \sum_{P=0}^{N_{\rm pix}-1} \qquad (7)$$

$$\times \left[\mu_l^*(P)T_{nkp}^l(P) + \mu_b^*(P)T_{nkp}^b\right] W_P,$$

$$s_{nkp} = \sum_{P=0}^{N_{\rm pix}-1} \qquad (8)$$

$$\times \left[\mu_l^*(P)\, S_{nkp}^l(P) + \mu_b^*(P)S_{nkp}^b\right] W_P,$$

in which the weights in the HEALPix and ECP schemes are, respectively,

$$W_P = \frac{4\pi}{N_{\rm pix}},$$

$$W_P = \frac{2\pi^2}{N_{\rm pix}} \cos b_P.$$

In the HEALPix scheme, the rms errors of these coefficients can be calculated from the formulas

$$\sigma_{v_{nkp}} = \sqrt{\frac{\sum_{P=0}^{N_{\rm pix}-1} \varepsilon_{V^*}^2(P)}{N_{\rm pix} - N_v}}, \qquad (9)$$

$$\sigma_{t_{nkp}} = \sigma_{s_{nkp}} \qquad (10)$$

$$= \sqrt{\frac{\sum_{P=0}^{N_{\rm pix}-1}\left[\varepsilon_{\mu_l^*}^2(P) + \varepsilon_{\mu_b^*}^2(P)\right]}{N_{\rm pix} - N_t - N_s}},$$

where N_v, N_t, and N_s are, respectively, the total numbers of calculated coefficients v_{nkp}, t_{nkp}, and s_{nkp}, while $\varepsilon_{V^*}(P)$, $\varepsilon_{\mu_l^*}(P)$, and $\varepsilon_{\mu_b^*}(P)$ are the residuals of the corresponding components of Eq. (1) calculated for the zone centers. In the ECP scheme, the squares of the residuals should be additionally multiplied by $\cos b_P$.

The number of decomposition terms can be chosen from the condition that the residues in the velocity field components after the subtraction of the statistically significant harmonics from them behave as random numbers (Brosche 1966).

As applied to the VSF method, both pixelization schemes have their advantages and disadvantages. An indubitable advantage of HEALPix over ECP is that the HEALpix fields are equal in area, while the areas of the spherical trapeziums in ECP decrease as the poles are approached. At the same time, the number of pixels lying on each parallel decreases in the polar zones are the poles are approached in HEALPix and is always constant in ECP. For this reason, the upper limit for the index k in the series of indices of the spherical functions n, k, p is always fixed in ECP, in contradistinction to HEALPix where it decreases rapidly as the poles are approached. In turn, because of aliasing, the high-frequency oscillations available in the data and reliably detected in the equatorial zone can manifest themselves in the polar zones at a different frequency.

It should also be noted that certain fields may turn out to be empty after data pixelization on the sphere. A significant number of empty fields can again generate false harmonics. This situation is well known in the theory and practice of unequally spaced time series analysis, where a nonuniform distribution of observation times generates the so-called "dirty" spectra. The special algorithms developed for cleaning such spectra (Roberts 1987; Vityazev 2001) can also be used in the VSF method.

Taking into account these circumstances, below we preferred to use a breakdown of the celestial sphere according to the ECP scheme with $M = 24$ and $N = 18$. This choice is justified by the great simplicity of the pixelization and a constant step of zone breakdown in longitude. Reducing the areas of the spherical trapeziums in the polar zones introduces no tangible errors into the results, since the bulk of the stars in the catalogs we used are contained in the Galactic equatorial zone.

Thus, the practical implementation of the VSF method consists of the following steps:

(1) *Elimination of the effects of solar motion among the stars from the radial velocities and proper motions*. Since the VSF decomposition coefficients responsible for the solar motion include the stellar parallaxes, it would be reasonable to determine and eliminate the solar motion from the stellar proper motions and radial velocities before the determination of other coefficients. If the stellar parallaxes are known, then this problem can be solved by solving the Airy–Kovalsy equations by the least-squares

Table 3. Parameters of the Ogorodnikov–Milne model adopted for the generation of test catalogs

U	V	W	ω_3	M_{12}^+	M_{11}^+
km s^{-1}			km s^{-1} kpc^{-1}		
10.0	20.0	8.0	−15.0	12.0	5.0

Table 4. Experiment 2. The VSF decomposition coefficients obtained for the model proper motions of Hipparcos stars in the range of distances 200–300 pc with a noise component of 10 km s^{-1} kpc^{-1} and the kinematic parameters calculated from them

t_{101}	-41.5 ± 1.2	s_{101}	0.0 ± 1.2	ω_1	0.2 ± 0.4
t_{110}	-0.4 ± 1.2	s_{110}	0.0 ± 1.2	ω_2	0.2 ± 0.4
t_{111}	0.5 ± 1.2	s_{111}	-0.1 ± 1.2	ω_3	-14.4 ± 0.4
t_{201}	0.3 ± 1.2	s_{201}	-1.5 ± 1.2	M_{12}^+	11.7 ± 0.6
t_{210}	2.4 ± 1.2	s_{210}	-0.3 ± 1.2	M_{13}^+	0.0 ± 0.5
t_{211}	0.6 ± 1.2	s_{211}	-0.1 ± 1.2	M_{23}^+	-0.1 ± 0.5
t_{220}	-0.8 ± 1.2	s_{220}	26.1 ± 1.2	M_{11}^*	4.6 ± 1.1
t_{221}	1.5 ± 1.2	s_{221}	5.2 ± 1.2	M_{33}^*	1.1 ± 1.1
t_{301}	1.1 ± 1.2	s_{301}	2.2 ± 1.2	–	–

Table 5. Experiment 2. The VSF decomposition coefficients obtained from the test OSACA catalog for stars in the range of distances 200–300 pc with a noise component of 10 km s^{-1} kpc^{-1}

v_{001}	7.3 ± 0.8	–	–	–	–
v_{101}	0.0 ± 0.8	t_{101}	-45.7 ± 3.1	s_{101}	-0.1 ± 3.1
v_{110}	-0.2 ± 0.8	t_{110}	0.3 ± 3.1	s_{110}	0.4 ± 3.1
v_{111}	0.0 ± 0.8	t_{111}	5.0 ± 3.1	s_{111}	-0.3 ± 3.1
v_{201}	-2.8 ± 0.8	t_{201}	7.3 ± 3.1	s_{201}	-0.7 ± 3.1
v_{210}	0.2 ± 0.8	t_{210}	3.0 ± 3.1	s_{210}	-5.1 ± 3.1
v_{211}	-0.5 ± 0.8	t_{211}	-8.3 ± 3.1	s_{211}	1.2 ± 3.1
v_{220}	20.3 ± 0.8	t_{220}	5.6 ± 3.1	s_{220}	24.9 ± 3.1
v_{221}	4.5 ± 0.8	t_{221}	5.7 ± 3.1	s_{221}	8.1 ± 3.1
v_{301}	-0.3 ± 0.8	t_{301}	5.8 ± 3.1	s_{301}	-2.7 ± 3.1

Table 6. Experiment 2. The kinematic parameters calculated using the coefficients from Table 5

From t_j and s_j		From v_j	
ω_1	1.7 ± 1.1	–	–
ω_2	0.1 ± 1.1	–	–
ω_3	-15.6 ± 1.1	–	–
M_{12}^+	11.1 ± 1.4	M_{12}^+	11.1 ± 0.4
M_{13}^+	0.5 ± 1.4	M_{13}^+	-0.3 ± 0.4
M_{23}^+	-2.3 ± 1.4	M_{23}^+	0.1 ± 0.4
$M_{11}^+ - M_{22}^+$	7.3 ± 2.7	M_{11}^+	5.4 ± 0.5
		M_{22}^+	0.5 ± 0.5
$M_{33}^* - M_{22}^+$	3.1 ± 2.7	M_{33}^+	0.3 ± 0.5

method. If the effects of solar motion were determined and eliminated, then the problem is subsequently reduced only to analyzing the rigid-body rotation velocity components and the velocity field deformation tensor elements. Otherwise, the VSF method will determine the components of the vector of solar motion to within a constant factor equal to the mean value of the parallaxes from the drawn sample of stars;

(2) *Data pixelization on the sphere.* At this step, the stellar radial velocities and proper motions are averaged over the spherical trapeziums obtained by uniformly dividing the Galactic equator and the meridian semicircle into M and N parts, respectively.

(3) *Calculation of the VSF decomposition coefficients v_j, t_j, s_j of the residual velocity field.* Eqs. (6)–(8) are used for this purpose. The rms errors of these decompositions can be calculated from Eqs. (36) and (37).

(4) *Determination of the parameters of a specific kinematic model.* Once the decomposition coefficients $v_j \pm \sigma_{v_j}, t_j \pm \sigma_{t_j}$, and $s_j \pm \sigma_{s_j}$ have been determined, they can be related to the parameters of a specific physical model. For the Ogorodnikov–Milne models, these relationships are given in Table 2.

Analysis of the results of experiment 2 shows that the VSF method is stable against random errors. Their action manifests itself in the rms errors of the coefficients. The estimates of the coefficients for the Ogorognikov–Milne model show no significant biases. The results based on the OSACA catalog are less accurate in random terms because of not only the smaller number of stars but also their highly nonuniform distribution over the celestial sphere (Fig. 1). As a result, the fields into which the stars fall contain significantly differing numbers of stars compared to the Hipparcos catalog (Fig. 2).

(5) *Analysis of the out-of-model decomposition coefficients.* As follows from Table 1, the Ogorodnikov–Milne (or Oort–Lindblad) model is completely described by the decomposition coefficients v_j and s_j up to $n \leq 2$ and by the coefficients t_j up to $n \leq 1$. All of the remaining decomposition terms with significant coefficients define the systematic components of the stellar velocity field that do not enter the standard models. Establishing the physical meaning of these harmonics is a separate problem that is basically reduced to constructing a new kinematic model.

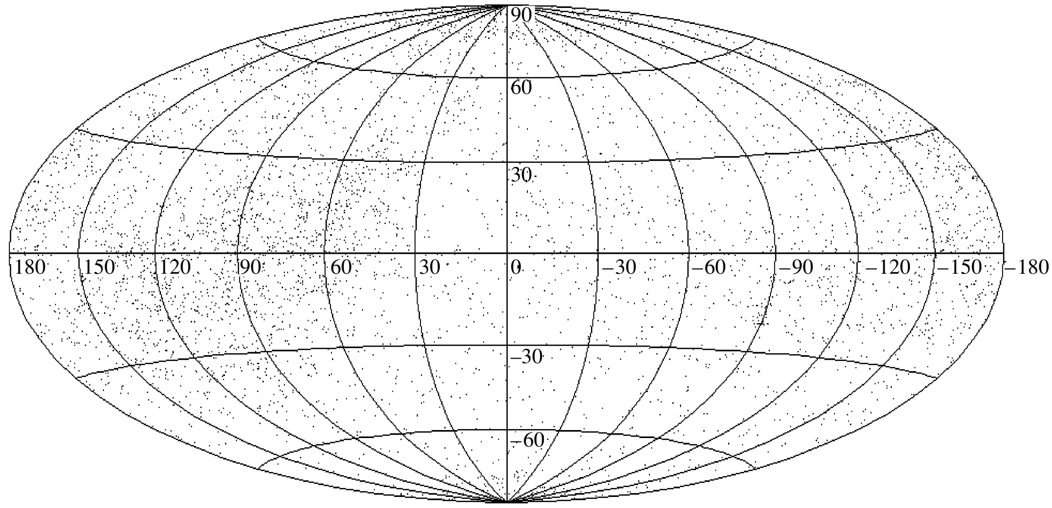

Fig. 1. Distribution of OSACA stars over the celestial sphere for the range of distances 200–300 pc.

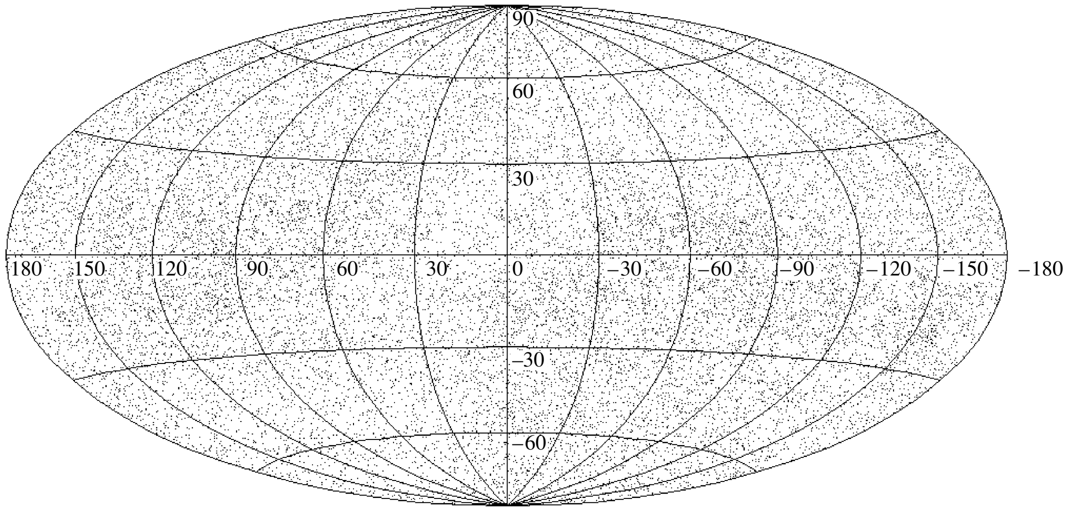

Fig. 2. Distribution of Hipparcos stars over the celestial sphere for the range of distances 200–300 pc.

NUMERICAL EXPERIMENTS

To test the capabilities of the VSF method, we carried out numerical experiments using two samples of stars in the range of distances 200–300 pc. The first and second samples consisted of Hipparcos and OSACA (Bobylev et al. 2006) stars, respectively. We assigned model proper motions to these stars in the former case and proper motions and radial velocities in the latter case. As a model stellar velocity field, we used Eqs. (3), (4), and (5) with the parameters from Table 3. The remaining parameters were taken to be zero.

Subsequently, three experiments were carried out for each sample:

(1) The VSF decomposition coefficients were determined directly from the model data.

(2) A large random component with a dispersion of 10 km s^{-1} kpc^{-1} was introduced into the model proper motions and radial velocities.

(3) A systematic component of the form

$$\sum_{j=9}^{15} (\mathbf{T}_j(l,b) + \mathbf{S}_j(l,b))$$

with an amplitude of 50 km s^{-1} kpc^{-1} was introduced into the model proper motions.

The results of the first experiment turned out to be in complete agreement with the theoretical predictions of Table 1. The rms errors of all coefficients were zero.

The VSF decomposition coefficients in the second experiment are given in Table 4 for the Hipparcos

Table 7. Experiment 3. The VSF decomposition coefficients obtained for stars at distances 200–300 pc from the Hipparcos catalog distorted by a systematic component

t_{101}	-43.6 ± 0.2	s_{101}	-0.3 ± 0.2
t_{110}	-0.1 ± 0.2	s_{110}	-0.7 ± 0.2
t_{111}	-0.3 ± 0.2	s_{111}	-0.4 ± 0.2
t_{201}	-0.1 ± 0.2	s_{201}	-3.2 ± 0.2
t_{210}	0.0 ± 0.2	s_{210}	-0.1 ± 0.2
t_{211}	-0.3 ± 0.2	s_{211}	-0.7 ± 0.2
t_{220}	-0.2 ± 0.2	s_{220}	27.0 ± 0.2
t_{221}	0.3 ± 0.2	s_{221}	5.1 ± 0.2
t_{301}	49.2 ± 0.2	s_{301}	49.3 ± 0.2
t_{310}	49.4 ± 0.2	s_{310}	48.7 ± 0.2
t_{311}	49.1 ± 0.2	s_{311}	48.8 ± 0.2
t_{320}	49.2 ± 0.2	s_{320}	49.2 ± 0.2
t_{321}	48.8 ± 0.2	s_{321}	48.9 ± 0.2
t_{330}	48.6 ± 0.2	s_{330}	48.6 ± 0.2
t_{331}	48.9 ± 0.2	s_{331}	48.5 ± 0.2

Table 8. Experiment 3. The kinematic parameters obtained by the standard method and the VSF method from a model velocity field distorted by a systematic component. The sample of stars in the range of distances 200–300 pc from Hipparcos

Parameter	LS[a]	VSF[b]
ω_1	0.5 ± 0.1	-0.1 ± 0.1
ω_2	0.3 ± 0.1	0.0 ± 0.1
ω_3	-17.6 ± 0.1	-15.1 ± 0.1
M^+_{12}	14.4 ± 0.3	12.0 ± 0.1
M^+_{13}	-0.4 ± 0.4	-0.3 ± 0.1
M^+_{23}	-3.7 ± 0.4	-0.1 ± 0.1
M^*_{11}	4.5 ± 0.8	4.9 ± 0.2
M^*_{33}	-2.6 ± 0.8	-0.2 ± 0.2

[a] The least-squares method.
[b] The VSF method.

catalog and Table 5 for the OSACA catalog. Tables 4 and 6 also present the kinematic parameters of the Ogorodnikov–Milne model calculated using the VSF decomposition.

The results of the third experiment for Hipparcos stars are presented in Table 7. We see that the introduced systematic component is completely isolated and distorts the low-frequency harmonics only slightly. The result obtained in the third experiment and presented in Table 8 is revealing. Here, we gave the parameters of the Ogorodnikov–Milne model obtained by the standard least-squares (LS) method directly from stars without using any data pixelization and by the VSF method using pixelization. We see that the VSF method yields a more accurate and reliable result, but the most important thing is that the standard LS method cannot detect any out-of-model systematics in the stellar proper motions (and radial velocities) *in principle*, although the "power of the signal" that is not described by the Ogorodnikov–Milne model exceeds significantly the contribution of this model for the parameters specified in Table 3. For a larger number of stars, we will obtain even formally small rms errors of the parameters, although *the LS method does not reveal the main effect in these artificial proper motions in any way*. It can be said that the LS method does not distinguish stochastic noise from systematic one. In contrast, the VSF method allows a more sophisticated analysis of the observational material to be performed by obtaining a "kinematic spectrum" of the systematic effects in the stellar proper motions and radial velocities whose subsequent investigation can point to the validity of a particular kinetic model for stellar motions.

The main conclusions drawn from the results of our numerical experiments are the following:

— in the cases where we are confident that, with the exception of white noise, there is no other systematic information than that described by the adopted model in the data under study, the VSF method has no particular advantages over the standard method based on the joint solution of the main kinematic equations by least squares;

— in the situations where, apart from random noise, systematic components that are not described by the model are present in our data, the VSF method shows clear advantages over the standard LS method, since it not only detects "out-of-model" components but also protects the model parameters from their influence.

— The VSF decomposition method makes it possible to test the stellar radial velocities and proper motions for compatibility. This is important, since these data are obtained by astrophysical and astrometric methods and can have different systematic errors, which will not be detected when the main kinematic equations are jointly solved by the LS method.

USING THE VSF METHOD TO STUDY REAL STELLAR VELOCITY FIELDS

In this section, we apply the VSF method to the proper motions of all Hipparcos stars and to the

Table 9. VSF decomposition coefficients of the proper motions for Hipparcos stars and parameters of the Ogorodnikov–Milne model calculated from them

Range, pc	50–100	100–200	200–300	300–400	400–600
Mean distance, pc	76	148	246	346	484
Number of stars	15 935	31 803	21 388	12 646	11 720
t_{101}	-43.5 ± 12.4	-29.4 ± 4.1	-38.8 ± 3.4	-40.5 ± 3.5	-43.0 ± 3.7
t_{110}	-30.9 ± 12.4	-18.6 ± 4.1	-5.5 ± 3.4	-2.4 ± 3.5	-3.2 ± 3.7
t_{111}	-12.6 ± 12.4	0.3 ± 4.1	-0.4 ± 3.4	1.9 ± 3.5	-3.4 ± 3.7
t_{201}	-7.9 ± 12.4	7.4 ± 4.1	6.7 ± 3.4	3.5 ± 3.5	7.5 ± 3.7
t_{210}	17.0 ± 12.4	11.8 ± 4.1	0.5 ± 3.4	-9.0 ± 3.5	-8.7 ± 3.7
t_{211}	-28.0 ± 12.4	6.6 ± 4.1	14.4 ± 3.4	20.0 ± 3.5	21.6 ± 3.7
t_{220}	-1.8 ± 12.4	6.6 ± 4.1	0.5 ± 3.4	3.1 ± 3.5	-3.7 ± 3.7
t_{221}	17.2 ± 12.4	5.7 ± 4.1	1.4 ± 3.4	-1.5 ± 3.5	1.1 ± 3.7
t_{301}	7.8 ± 12.4	-7.5 ± 4.1	-4.7 ± 3.4	9.0 ± 3.5	1.2 ± 3.7
t_{310}	20.3 ± 12.4	5.2 ± 4.1	5.4 ± 3.4	-0.1 ± 3.5	0.9 ± 3.7
t_{311}	-11.3 ± 12.4	0.9 ± 4.1	3.9 ± 3.4	-2.9 ± 3.5	-6.1 ± 3.7
t_{320}	-2.7 ± 12.4	-5.2 ± 4.1	2.7 ± 3.4	-2.6 ± 3.5	-2.9 ± 3.7
t_{321}	21.5 ± 12.4	-4.8 ± 4.1	2.5 ± 3.4	3.8 ± 3.5	0.0 ± 3.7
t_{330}	12.6 ± 12.4	1.5 ± 4.1	2.7 ± 3.4	-1.0 ± 3.5	-1.1 ± 3.7
t_{331}	4.4 ± 12.4	-2.9 ± 4.1	-3.5 ± 3.4	3.9 ± 3.5	-1.1 ± 3.7
s_{101}	-0.2 ± 12.4	0.1 ± 4.1	0.0 ± 3.4	0.0 ± 3.5	-0.1 ± 3.7
s_{110}	0.8 ± 12.4	0.3 ± 4.1	0.1 ± 3.4	0.0 ± 3.5	0.0 ± 3.7
s_{111}	0.3 ± 12.4	-0.1 ± 4.1	-0.1 ± 3.4	-0.1 ± 3.5	0.0 ± 3.7
s_{201}	-1.1 ± 12.4	-3.4 ± 4.1	2.9 ± 3.4	2.4 ± 3.5	5.3 ± 3.7
s_{210}	-19.6 ± 12.4	-5.0 ± 4.1	2.3 ± 3.4	0.9 ± 3.5	-2.7 ± 3.7
s_{211}	-22.2 ± 12.4	-6.0 ± 4.1	2.5 ± 3.4	-2.6 ± 3.5	-1.1 ± 3.7
s_{220}	36.5 ± 12.4	20.0 ± 4.1	31.8 ± 3.4	28.4 ± 3.5	25.8 ± 3.7
s_{221}	21.4 ± 12.4	-4.5 ± 4.1	-6.9 ± 3.4	-10.1 ± 3.5	-6.2 ± 3.7
s_{301}	-7.0 ± 12.4	2.8 ± 4.1	-0.5 ± 3.4	-8.1 ± 3.5	-9.2 ± 3.7
s_{310}	7.9 ± 12.4	-4.6 ± 4.1	-18.0 ± 3.4	-12.7 ± 3.5	-15.7 ± 3.7
s_{311}	15.1 ± 12.4	-2.8 ± 4.1	-0.6 ± 3.4	-4.8 ± 3.5	4.7 ± 3.7
s_{320}	-26.6 ± 12.4	2.6 ± 4.1	-2.5 ± 3.4	3.9 ± 3.5	-5.2 ± 3.7
s_{321}	11.1 ± 12.4	0.3 ± 4.1	-0.7 ± 3.4	1.2 ± 3.5	-5.1 ± 3.7
s_{330}	39.7 ± 12.4	15.9 ± 4.1	-0.8 ± 3.4	1.2 ± 3.5	-2.3 ± 3.7
s_{331}	44.0 ± 12.4	-2.3 ± 4.1	-4.1 ± 3.4	-0.5 ± 3.5	-2.5 ± 3.7
ω_1	-4.4 ± 4.3	0.1 ± 1.4	-0.1 ± 1.2	0.6 ± 1.2	-1.2 ± 1.3
ω_2	-10.7 ± 4.3	-6.4 ± 1.4	-1.9 ± 1.2	-0.8 ± 1.2	-1.1 ± 1.3
ω_3	-15.1 ± 4.3	-10.2 ± 1.4	-13.4 ± 1.2	-14.0 ± 1.2	-14.9 ± 1.3
M^*_{11}	19.1 ± 11.1	-4.1 ± 3.7	-6.2 ± 3.1	-9.0 ± 3.1	-5.5 ± 3.3
M^*_{33}	8.7 ± 11.1	-4.6 ± 3.6	-0.8 ± 3.0	-2.7 ± 3.1	1.3 ± 3.3
M^+_{12}	16.3 ± 5.6	8.9 ± 1.8	14.2 ± 1.5	12.6 ± 1.5	11.5 ± 1.6
M^+_{13}	-9.9 ± 5.5	-2.7 ± 1.8	1.1 ± 1.5	-1.2 ± 1.5	-0.5 ± 1.6
M^+_{23}	-8.7 ± 5.5	-2.2 ± 1.8	1.0 ± 1.5	0.4 ± 1.5	-1.2 ± 1.6

Note. The units of measurement are km s^{-1} kpc^{-1}.

Table 10. VSF decomposition coefficients of the radial velocities for OSACA stars and parameters of the Ogorodnikov–Milne model calculated from them

Range, pc	100–200	200–300	300–400	400–600
Mean distance, pc	143	243	348	479
Number of stars	10 696	4830	2611	2297
v_{001}	8.6 ± 7.3	-14.8 ± 9.2	6.3 ± 13.1	-5.1 ± 13.8
v_{101}	-0.4 ± 7.3	0.0 ± 9.3	2.3 ± 13.6	1.8 ± 14.5
v_{110}	0.0 ± 7.3	0.1 ± 9.2	-3.0 ± 12.8	-1.8 ± 13.4
v_{111}	0.1 ± 7.3	0.1 ± 9.2	2.0 ± 12.8	-5.6 ± 13.4
v_{201}	-1.4 ± 7.3	-13.8 ± 9.3	-10.1 ± 13.5	11.6 ± 14.2
v_{210}	4.0 ± 7.3	-2.4 ± 9.3	5.7 ± 13.3	-6.4 ± 14.0
v_{211}	2.8 ± 7.3	6.5 ± 9.2	20.8 ± 13.3	35.0 ± 14.1
v_{220}	20.0 ± 7.3	21.3 ± 9.2	6.5 ± 12.7	13.7 ± 13.3
v_{221}	-0.1 ± 7.3	-11.5 ± 9.2	-5.8 ± 12.7	-30.2 ± 13.2
v_{301}	-0.6 ± 7.3	-3.1 ± 9.3	11.9 ± 13.2	3.1 ± 13.9
v_{310}	-8.8 ± 7.4	-23.0 ± 9.2	-41.4 ± 13.5	-51.1 ± 14.1
v_{311}	5.5 ± 7.4	-9.7 ± 9.3	-1.4 ± 13.5	-25.8 ± 14.2
v_{320}	-17.4 ± 7.3	2.0 ± 9.2	2.5 ± 13.0	-6.0 ± 13.9
v_{321}	3.3 ± 7.3	-13.6 ± 9.2	-1.3 ± 13.0	-9.0 ± 13.6
v_{330}	-7.5 ± 7.3	5.8 ± 9.2	-11.7 ± 12.7	1.5 ± 13.2
v_{331}	9.4 ± 7.3	0.2 ± 9.2	-7.5 ± 12.6	-21.5 ± 13.1
M_{11}^{+}	2.8 ± 5.1	-6.1 ± 6.3	1.8 ± 8.9	-21.5 ± 9.3
M_{22}^{+}	2.9 ± 5.1	6.4 ± 6.3	8.1 ± 8.9	11.3 ± 9.3
M_{33}^{+}	3.3 ± 5.1	4.5 ± 6.4	8.1 ± 9.2	-8.7 ± 9.7
M_{12}^{+}	10.9 ± 4.0	11.6 ± 5.0	3.6 ± 6.9	7.5 ± 7.2
M_{13}^{+}	1.6 ± 4.0	3.5 ± 5.0	11.3 ± 7.2	19.1 ± 7.7
M_{23}^{+}	2.2 ± 4.0	-1.3 ± 5.1	3.1 ± 7.2	-3.5 ± 7.6

Note. The units of measurement are km s^{-1} kpc^{-1}.

Hipparcos stars with known radial velocities. Our analysis was based on samples of stars in narrow ranges of distances. Before calculating the sought-for coefficients, we eliminated the effects of solar motion, whose parameters were determined from the Airy–Kovalsky equations by taking into account the individual stellar parallaxes, from the stellar proper motions and radial velocities.

The results obtained from the stellar proper motions are presented in Table 9. Comparison of the values in this table with those in Table 1 leads us to several conclusions.

For nearby stars with distances 50–100 pc, the coefficients are determined with significant errors (the errors are even larger for stars closer than 50 pc). This suggests that at small distances the peculiar components dominate in the stellar proper motions, while the systematic component is barely traceable.

Nevertheless, we can note the high values and significance of the coefficients t_{101} and t_{110}, suggesting that the rotation axis of the system of nearby stars is inclined to the Galactic plane (Tsvetkov 2006). The coefficients s_{211}, s_{220}, and s_{221}, which show deformations in other planes and expansion of the system of nearby stars, are also large.

As the distance increases, the accuracy of determining the coefficients improves significantly (the effect of the peculiar stellar velocities is reduced). Out of the above coefficients, only t_{101} and s_{220}, which describe the "classical" plane Galactic rotation, remain statistically significant. At the same time, the coefficients t_{211} and s_{310}, whose physical meaning cannot be interpreted in terms of linear models, become statistically significant.

In addition to the analysis of stellar proper motions, we decomposed 35 847 radial velocities of OSACA stars into VSFs (Table 10). Analysis of this table leads us to similar conclusions: in all samples, not only the coefficient v_{220}, which is generated by the parameter M_{12}, but also the coefficient v_{310}, which is not predicted by the Ogorodnikov–Milne model, is statistically significant.

It should be noted that the relative accuracy of determining the radial velocities is lower than the accuracy of determining the proper motions. In addition, the nonuniformity of the distribution of OSACA stars over the celestial sphere increases greatly with distance. This gives rise to a large number of empty fields and, as a result, leads to the loss of pixelization grid uniformity. Furthermore, the presence of random errors in the stellar parallaxes also causes the result to become less reliable, since, apart from an incorrect determination of whether a star belongs to a particular range of distances (which is also the case when the proper motions are analyzed), we decompose not the radial velocity itself but $V_r\pi$; as a result, the parallax errors penetrate into the coefficients V_j to be determined. For these reasons, both VSF decomposition coefficients of the velocity field and parameters of the Ogorodnikov–Milne model are determined from the radial velocities much more poorly than from the proper motions.

INTERPRETATION OF THE RESULTS IN TERMS OF THE GENERALIZED OORT MODEL

To interpret the revealed "out-of-model" terms of the velocity field, we should consider models different from the linear Ogorodnikov–Milne model used. As the first step, it is reasonable to consider a second-order Ogorodnikov–Milne model (Edmondson 1937). However, this model predicts the existence of a large number of harmonics that are not observed in the real data. Therefore, for our analysis, we chose a simpler model constructed on "generalized Oort formulas" (Ogorodnikov 1965; Bobylev et al. 2007). This is also a second-order model, but it considers the variations in the angular velocity of rotation of the stellar system only toward the Galactic center. The equations of this model are

$$\mathcal{K}\mu_l \cos b = (U/r)\sin l - (V/r)\cos l + A\cos b\cos 2l + B\cos b - rF\cos^2 b\cos^3 l - rG(3\cos^2 b\cos l - \cos^2 b\cos^3 l), \quad (11)$$

$$\mathcal{K}\mu_b = (U/r)\cos l\sin b + (V/r)\sin l\sin b - W/r\cos b - A\cos b\sin b\sin 2l + rF\cos^2 b\sin b\sin l\cos^2 l + rG\cos^2 b\sin b\sin^3 l - K\cos b\sin b, \quad (12)$$

Table 11. Contribution from the parameters of the generalized Oort model to the VSF decomposition coefficients of the velocity field

Coefficient	Value
v_{001}	$2.36K$
v_{101}	$-2.05W/\langle r\rangle$
v_{110}	$-2.05V/\langle r\rangle - 0.41F\langle r\rangle - 1.23G\langle r\rangle$
v_{111}	$-2.05U/\langle r\rangle$
v_{201}	$-1.06K$
v_{220}	$1.83A$
v_{310}	$0.11F\langle r\rangle + 0.33G\langle r\rangle$
v_{330}	$-0.42F\langle r\rangle + 0.42G\langle r\rangle$
t_{101}	$2.89B$
t_{211}	$-0.75F\langle r\rangle - 2.24G\langle r\rangle$
s_{101}	$-2.89W/\langle r\rangle$
s_{110}	$-2.89V/\langle r\rangle - 1.15F\langle r\rangle - 3.47G\langle r\rangle$
s_{111}	$-2.89U/\langle r\rangle$
s_{201}	$-1.29K$
s_{220}	$2.24A$
s_{310}	$0.12F\langle r\rangle + 0.38G\langle r\rangle$
s_{330}	$-0.49F\langle r\rangle + 0.49G\langle r\rangle$

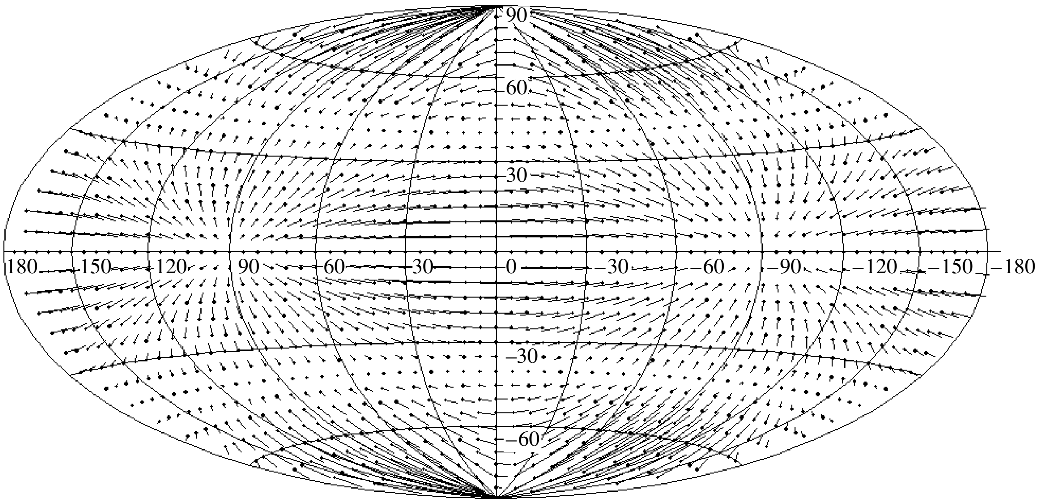

Fig. 3. View of the revealed systematic component in the proper motions.

$$V_r/r = -(U/r)\cos b\cos l - (V/r)\cos b\sin l - (W/r)\sin b + A\cos^2 b\sin 2l - rF\cos^3 b\sin l\cos^2 l - rG\cos^3 b\sin^3 l + K\cos^2 b. \quad (13)$$

where $A = 0.5R_0\omega_0'$ and $B = 0.5R_0\omega_0' + \omega_0$ are the Oort parameters (R_0 is the distance to the Galactic center, ω_0 is the angular velocity of Galactic rotation), F and G are the second-order Oort parameters, $F = 0.5R_0\omega_0''$ and $G = A/R_0$.

We decomposed Eqs. (11)–(13) into a set of VSFs. The results are presented in Table 11. Analysis of this table shows that in the presence of strong second-order effects and identical signs of the parameters F and G, one might expect the appearance of statistically significant coefficients t_{211}, s_{310}, and v_{310}. The values of s_{310} and v_{310} should coincide, while their signs should be opposite to the sign of t_{211}. The coefficient t_{211} should exceed s_{310} and v_{310} in absolute value. When the second-order Oort parameters are equal in order of magnitude, the coefficients s_{330} and v_{330} will probably be statistically insignificant.

Comparing these results with the coefficients t_{211}, s_{310}, v_{310} as well as s_{330}, v_{330} taken from Tables 9 and 10, we can assert that the generalized Oort model describes satisfactorily the kinematics of stars at heliocentric distance of more than 300–400 pc on a qualitative level. However, there are also quantitative discrepancies: it follows from Table 11 that the coefficient t_{211} should exceed s_{310} by a factor of 5.8 in absolute value, but it actually exceeds the latter only by a factor of 1.5. In addition, although s_{310} and v_{310} coincide in sign, they differ approximately by a factor of 3. Thus, a search for more accurate stellar kinematic models of the solar neighborhood is needed.

CONCLUSIONS

We detected "out-of-model" components in the proper motions of Hipparcos stars (in particular, $\mathbf{S}_{310}$) previously (Vityazev and Shuksto 2004, 2005). An important result of this study is the detection of out-of-model components in the stellar radial velocities as well. Indeed, we were able to show that the real velocity field of the stars at heliocentric velocities of 200–600 pc contains the component

$$\mathbf{U}(l,b) = v_{310}\mathbf{V}_{310} + s_{310}\mathbf{S}_{310} + t_{211}\mathbf{T}_{211} \quad (14)$$

that does not enter into the standard linear model. As was shown above, the appearance of such a component could be expected only in more complex models. It should be noted that when using the traditional methods of analysis based on the least-squares solution of Eqs. (3)–(5), we would obtain quite reliable parameters of the standard model, but the entire complexity of the true kinematics of the solar neighborhood would remain unrevealed. The manifestation of the systematic component (14) in the stellar proper motions is shown in Fig. 3.

This shows the advantages of the VSF method in analyzing the stellar velocity fields and interpreting the results in terms of various models.

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