The new method of Friedmann equations solving and spacetime without events horizons

© A.V. Yurov [1, 2], V.A. Astashenok [1, 3]

[1] I. Kant Russian State University, Theoretical Physics Department, Kaliningrad, Russia
[2] E-mail: artyom_yurov@mail.ru, [3] E-mail: artyom.art@gmail.com

Abstract: The new method of constructing of exact solutions of cosmological equations is developed. The method is based on representation of Friedmann equations in form of linear differential second order equation. It is shown that in some cases the appearing classes of solutions describe spacetime without events horizons, i.e. any geodesic circumnavigate the universe an infinite number of times as the future c-boundary is approached. Probably in this case the self-consistensy cosmological model can be construct.

1. Introduction: the self-consistency condition

Generally the discovery of contradictions between well-known physical rules and data of cosmological observations is explained as evidence of existence of undiscovered fundamental principles. However there is another approach, namely one regard the basic physical principles as already established and use of mentioned contradictions for selection of cosmological models without of this oppositions (the self-consistency condition). In this case the study of global universe evolution is very important.

As follows from the recent observations (see [1], [2]) our universe suffers the accelerated expansion [3], [4]. As for now, the one of the probable cause of such expansion lies in nonzero cosmological constant. If this is really the case, then the future dynamics of observable universe is confined within the particles horizon $R_h = c/H$ and, as such, leads to problems with formulation of a fundamental physical theory (like the string theory or hypothetical M-theory) in a finite volume [5]. Seemingly, the consistent mathematical description is possible only for $R_h = \infty$.

This argument was used in [6] as a point in support of cosmological models with phantom energy in “holographic” form [7]. Author [6] showed that in this case it is possible that $R_h = \infty$ and therefore, the fundamental theory may be formulated without contradictions.

On the other hand, the phantom energy is experimentally indistinguishable from vacuum energy, but definitely distinct from any other known form of matter and looks very exotic. It is reasonable, then, to consider dark energy to be vacuum energy [8].

Furthermore the unlimited expansion of universe leads to paradoxical conclusion: if the lifetime of the universe (with an observable expansion rate) will exceed the limit of 10^{60} years, the dominating observers will, as follows from [9], be the ones of a quantum fluctuations origin, which, of course, could hardly be called compatible with our observations. Moreover, the estimate for maximal lifetime in [9] sharply distinguish from estimates for lifetime τ of metastable dS-phase. According to [10], [11] $\tau < \mathrm{e}^{0.5\times 10^{123}}$ yr. In models contained KPV-instantons [12] this value can be decreased to $\tau < \mathrm{e}^{10^9}$ yr. However this result is enormously greater than 10^{60} years.

Therefore there are many arguments against the models of universe suffering the eternal expansion. The most probable scenarios would be those describing the contemporary expansion, being traced by the consequent contraction phase and the ``horizonless" collapse.

In this paper we present a new simple method of construction of infinite number of solutions of Freidmann equations from the already known ones. The method is based on representation of Friedmann equations in form of linear differential second order equation. In some cases the appearing classes of solutions describe the above-mentioned scenarios.

2. The method of linearization for Friedmann equations

Let us write the Friedmann equations as

$$\left(\frac{\dot a}{a}\right)^2 = \rho - \frac{k}{a^2}, \qquad 2\frac{\ddot a}{a} = -(\rho + 3p), \tag{1}$$

where $k = 0, \pm 1$. We use units with $8\pi G/3 = c = 1$. If the universe is filled with a self-acting and minimally coupled scalar field with Lagrangian

$$L = \frac{\dot{\phi}^2}{2} - V(\phi) = K - V, \tag{2}$$
then the energy density and pressure are
$$\rho = K + V, \qquad p = K - V,$$
therefore
$$V = \frac{1}{2}(\rho - p), \qquad K = \frac{1}{2}(\rho + p). \tag{3}$$
Our starting point is that the volume function $\psi = a^3$ satisfies a simple second-order differential equation ([**Ошибка! Источник ссылки не найден.**])
$$\ddot{\psi} = 9V\psi. \tag{4}$$
In (**Ошибка! Источник ссылки не найден.**) the potential V is represented as a function of time t. For simple forms of the potential one can find the general solution of (**Ошибка! Источник ссылки не найден.**), containing both the solution used for the construction of this potential and a lineary independent one as well. Substituting this general solution into the (**Ошибка! Источник ссылки не найден.**) one can calculate ρ and p. Then using (**Ошибка! Источник ссылки не найден.**) one gets the new potential $\tilde{V}$ such that $\tilde{V}(t) = V(t)$ but whose form is different from V if $\tilde{V}$ and V are represented as functions of ϕ: $\tilde{V}(\tilde{\phi}) \neq V(\phi)$.

Therefore the Friedmann equations admits the linearizing substitution and can be studied via different powerful mathematical methods which were developed for the linear differential equations. This is the reason why we call our approach the method of linearization. The crucial point of this paper is connected to the simple generalization of results above. More precisely, the following proposition is hold:

Proposition. Let $a = a(t)$ (with $p = p(t)$, $\rho = \rho(t)$) be the solution of (1) with $k = 0$. Then the function $\psi_n \equiv a^n$ is the solution of the Schrödinger equation
$$\frac{\ddot{\psi}_n}{\psi_n} = U_n, \tag{5}$$
with potential
$$U_n = n^2\rho - \frac{3n}{2}(\rho + p). \tag{6}$$

Remark 1. If the universe is filled with scalar field ϕ whose Lagrangian is (2) then the expression (6) will be
$$U_n = n(n-3)K + n^2 V.$$
In particular case $n = 3$ $U_3 = 9V(\phi)$ (see (4)). This particular case has been extensively studied in [13], [14].

Remark 2. If besides matter fields in the universe there is nonzero vacuum energy Λ the Eq. (5) takes the form
$$\frac{\ddot{\psi}_n}{\psi_n} = U_n - \lambda_n, \tag{7}$$
with $\lambda_n = -n^2\Lambda$.

In the case of general position, the solution of the equation (5) or (6) has the form
$$\Psi_n = c_1\psi_n + c_2\hat{\psi}_n, \tag{8}$$
where $\hat{\psi}_n$ is linearly independent counterpart of ψ_n:
$$\hat{\psi}_n(t) = \psi_n(t)\int\frac{dt'}{\psi_n^2(t')} \equiv \psi_n(t)\xi(t). \tag{9}$$
Equation (8) is enough to establish the following theorem:

Linearization Theorem. Let $a = a(t)$ be the solution of (1) with $k = 0$ and with ρ and p, given by (**Ошибка! Источник ссылки не найден.**). Then the two-parameter function $a_n = a_n(t; c_1, c_2)$:
$$a_n = a\left(c_1 + c_2\int\frac{dt}{a^{2n}}\right)^{1/n}, \tag{10}$$
will be solution of (1) with new energy density ρ_n and pressure p_n satisfying:
$$n^2\rho_n - \frac{3n}{2}(\rho_n + p_n) = n^2\rho - \frac{3n}{2}(\rho + p). \tag{11}$$
Another way to formulate this theorem is to say that the expression

$$U_n = \frac{n\left((n-1)\dot{a}^2 + a\ddot{a}\right)}{a^2},$$
is invariant with respect to transformation $a \to a_n$ with a_n defined by (10). We'll use the term "dressing" for the process of transformation of a triple $\{a, \rho, p\}$, with the resulting triple $\{a_n, \rho_n, p_n\}$ being referred to as the dressed one.

Remark 3. This theorem is valid for the case $k = 0$. If $k = \pm 1$ then this theorem will hold if and only if $n = 0,1$.

Remark 4. Let $a(0) = 0$, i.e. suppose that at $t = 0$ there exist an initial singularity. Lets assume that $a(t) \sim t^{\lambda}$ for $t \to 0$. One might easily verify that if $2n\lambda \le 1$ then $a_n(0) = 0$. Now let us choose $|c_1/c_2| >> |\xi_{max}|$ where $\xi = \xi(t)$ is the quantity from (10); ξ is a bounded function at the interval $0 < t < T$ and ξ_{max} is the maximal value of $\xi(t)$ at this interval. It can been seen that at a given time interval $a_n(t)$ behaves similar to $a(t)$ with any given precision rate.

Moreover, since both matter's density and pressure are expressed in terms of scale factor (and it's derivatives) explicitly (we remind here, that under the assumption the sign of curvature is already known): then we conclude that the observations have given the values of all basic characteristics of the universe. Can we say now that the further evolution of universe will be completely defined? No, we can't be sure that the "real" scale factor is $a(t)$. It can be $a_n(t)$ as well (note here, that possibility of such scenario, i.e. of indeterminacy of universe future has been previously noted by A.A. Starobinsky in [15]).

3. A toy self-consistent model

Let’s consider a simple example of constructing self-consistent model via method considering in previous section. For $U_n = \mu^2 t^2$ ($\mu = const$) the simplest solution of Eq. (6) is

$$\psi_n = C\exp(-\mu t^2/2) \tag{12}$$

It is easy to see that the evolution of scale factor is similar for various $n > 0$. Without loss of generality one consider the case $n = 1$. It is convenient to write the solution for scale factor as

$$a(t) = a_0 \exp(\mu(t_0^2 - t^2)/2) \tag{13}$$

where t_0 is the moment of observation in current time, $a_0 = 10^{28}$ cm is the current value of scale factor. The parameter μ can be expressed via t_0 and Hubble parameter $H_0 = 24.3\times10^{-19}$ s^{-1}: $\mu = -H_0/t_0$. The absence of events horizons in this model is obvious. The t_0 is free parameter so solution (13) can be matched to the data of astronomical observations which demand that

$$\frac{\ddot{a}(t_0)}{a(t_0)} = \frac{8\pi G}{3}\times\left(0.7\rho_c\right) = \frac{7H_0^2}{10}. \tag{14}$$

Using (13) we get

$$t_0 = -\frac{10}{3H_0}. \tag{15}$$

Lets assume that the age of universe T is $T = t_0 - t_{pl}$ at that $a(t_{pl}) = a_{pl} = 10^{-33}$ cm. The number of e-foldings is

$$N = \ln\frac{a_0}{a_{pl}} = 61\ln 10 \approx 140\,.$$

The age T obeys the equation:

$$3H_0^2T^2 + 20H_0T - 20N = 0,$$

that yields

$$T = \frac{2(\sqrt{25+15N} - 5)}{3H_0} \approx \frac{27.4}{H_0} = 356\,\text{Gyr}.$$

This is too large in comparison with 15 – 20 Gyr. However, formally speaking the age of such universe is infinite. We assumed that the origin of the universe is the moment when $a = 10^{-33}$ cm. This is natural choice but may be that choice is wrong. For example, one can assume that solution (13) describes the universe suffered hypothetical big trip (see [16], [17]). In this case the estimates of universe age depend from the value of $w_i = w(t_i) = p/\rho$ where t_i is the moment of big trip. For instance if $w_i = -0.9$ we have $T = t_0 - t_i \approx 18$ Gyr. It is clear that this is only interpretation.

In this model one can calculate the value of cosmological constant. If the universe is filled with a self-acting and minimally coupled scalar field ϕ with potential $V = V(\phi)$ and vacuum energy with density ρ_Λ. Then

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_\phi + \frac{3p_\phi}{c^2}\right) + \frac{8\pi G\rho_\Lambda}{3} = 4\mu^2 t^2 - 2\mu.$$

From this relation one yield that $\rho_\Lambda = -9H_0^2/(80\pi G) = -0.3\rho_c$. The cosmological constant is negative although model describes the current stage of acceleration. The form of potential is

$$V(\phi) = \frac{9H_0^2\phi^2}{20c^2}.$$

In other words we have scalar field describing the non-interacting scalar particles with extremely small mass $m \sim 0.3\times 10^{-60}$ g.

At first glance the significant cosmological acceleration in past should be put obstacles in the way of generation of the observable large-scale structure. However it is not so. If ρ_M is a density of baryon matter then

$$\rho_M = \rho_{M0}(1+z)^3. \tag{16}$$

Here ρ_{M0} is the current density of baryon matter ($\sim 10^{-31}$ g/cm^3), z is the value of red shift. The following constraint on the value of acceleration must be hold

$$\frac{3\ddot{a}(t_m)}{8\pi G a(t_m)} \le \rho_{M0}(1+z_m)^3 \tag{17}$$

where z_m corresponds to the moment t_m of the earliest galaxies forming. The up observable limit for z_m is ~10. Using (13), (15) it is easy to establish that condition (17) is not violated.

4. Spacetime without events horizons as a generalization of well-known solutions

The equation (10) allows one to construct the space-time without events horizon from well-known solutions. In order to show this let t_f be an instant, such that $a_n(t_f; c_1, c_2) = 0$ but $a(t_f) = a_f \neq 0$. In other words, suppose that $c_1 + c_2\xi_f = 0$, where $\xi_f = \xi(t_f)$. Then, for $t \to t_f$ (and $t < t_f$) one gets

$$a_n \sim a_f\left(c_1 + c_2\xi_f - c_2(t_f - t)\dot{\xi}_f\right)^{1/n} = \frac{\kappa}{a_f}(t_f - t)^{1/n}, \tag{18}$$

where $c_2 = -\kappa^2 < 0$. Integrating this equation for future directed radial null geodesics, $ds^2 = dt^2 - a_n^2 d\chi^2$ one will get

$$\Delta\chi \sim \frac{a_f}{\kappa}\int\frac{dt'}{(t_f - t)^{1/n}} \sim \frac{n a_f}{\kappa(1-n)}\lim_{t\to t_f}(t_f - t)^{(n-1)/n}.$$

It is easy to see that for $0 < n \le 1$ we will have $\Delta\chi = +\infty$ which shows that radial null geodesics circumnavigate the universe infinite number of times as the future c-boundary at $t = t_f$ is approached. By homogeneity and isotropy, we can conclude that all future endless timelike curves define the same c-boundary point.

In the case $n = 1$ one gets $\Delta\chi \sim -\log(t_f - t) \to +\infty$ as the final singularity is approached. This case is extremely interesting because (see Remark 3) when $n = 1$ one can use the dressing procedure to construct the exact solutions for the universes with $k = \pm 1$.

A. Simple generalization of $k = +1$ dust model

In the simplest dust case with $p = 0$ one can solve the system (1) to get

$$a = a_m \sin^2 \eta, \qquad 2\eta - \sin 2\eta = \frac{2t}{a_m},$$
$$\rho = \frac{1}{a_m^2 \sin^6 \eta}. \tag{19}$$
Using (10) for the case $n = 1$ one will obtain the general solution
$$a(t)_{gen} = c_1 a(t) + c_2 \hat{a}(t), \tag{20}$$
where $c_{1,2}$ are the arbitrary constants. It is possible to rewrite (20) in the form
$$a(t) \equiv a(t)_{gen} = A \sin\eta \sin(\delta - \eta), \tag{21}$$
with two arbitrary constants A and δ. This solution describes the universe being born from the initial singularity ($\eta_i = 0$, $t_i = 0$) and collapsing thereafter into the final singularity at $\eta_f = \delta$ or
$$t_f = \frac{1}{2\alpha}(2\delta - \sin 2\delta)$$
where we have introduced a new parameter α such that
$$2\eta - \sin 2\eta = 4\alpha t, \qquad \alpha = \frac{1}{2a_m}\sin^2\frac{\delta}{2};$$
hence,
$$a = \frac{a_m}{\sin^2(\delta/2)} \sin\eta \sin(\delta - \eta), \tag{22}$$
and the maximum value of $a = a_m$ will occur at $\eta = \delta/2$.

It is easy to see that upon the choice $\delta = \pi$ one gets the well known "dust solution" (19). In case of general position one shall choose $0 < \delta < \pi$. It can be seen that for $t << t_f$, $\delta = \pi - \varepsilon$ and $\varepsilon << 1$, (22) will behave similar to (19). But if $t \sim t_f$ then one gets something really different: a universe without events horizon. To show this lets consider
$$ds^2 = dt^2 - \frac{1}{4\alpha^2}\sin^2\eta \sin^2(\delta - \eta)\left[d\chi^2 + \sin^2\chi d\Omega_3^2\right]$$
Upon integration of the equation $ds^2 = 0$ (describing the future directed radial null geodesics) one gets:
$$\Delta\chi = 2\int_\eta^\delta \frac{\sin\eta}{\sin(\delta - \eta)} d\eta = +\infty. \tag{23}$$
We note that if $\delta = \pi$ then
$$\Delta\chi = 2(\pi - \eta) < \infty.$$
The result (23) shows that radial null geodesics circumnavigate the universe an infinite number of times as $t \to t_f$. This fact and the homogeneity+isotropy results in conclusion that (i) this universe has no event horizons and (ii) all future endless timelike curves define the same c-boundary point. At last,
$$\Delta\chi = 2\int_0^\eta \frac{\sin\eta}{\sin(\delta - \eta)} d\eta < +\infty.$$
One can show that all energy conditions are valid. For this let us point out that sum $\rho + 3p$ in general model is equal to the sum $\rho + 3p$ in starting model (see (11) for the case $n = 1$). This fact results in validity of strong energy condition for our model. Finally, from Friedmann equations one can see that density of energy is always positive at $k = 1$. By this property and the validity of the strong energy condition, the weak energy one will be satisfied automatically.

B. Generalization of a Lambda-radiation model in flat space

Let's consider the flat universe which has a positive vacuum energy Λ. Let's also assume that the universe is filled with the radiation. Solving system (1) we will obtain the initial solution for the scale factor
$$a = a_0 \sinh^{1/2}\theta, \qquad \theta = 2\sqrt{\Lambda} t,$$
$$\rho = \Lambda(1 + \sinh^{-2}\theta), \tag{24}$$

where a_0 is a positive constant. If $t > t_v = 2^{-1}\Lambda^{-1/2}\text{arcsinh}1$, the strong energy condition will necessarily be violated. Using (10) for the case $n = 1$ one can see that general solution can be written in the following form

$$a_{gen} \equiv a = a_0 \sinh^{1/2}\theta \ln \frac{\coth^{\varepsilon}\frac{\theta}{2}}{\delta}, \tag{25}$$

where δ is a positive constant. The parameter ε, introduced here, plays an important role in our reasonings. If $\varepsilon = 0$, then (25) will be equivalent to the initial solution. In the remaining cases one can without any loss of generality assume $\varepsilon = 1$. There will be three types of solutions. If $\delta < 1$ the universe will be open. This type of solutions has the following asymptotic behavior

$a \to -0.5 a_0 \ln\delta \exp(\sqrt{\Lambda} t), \qquad t \to \infty.$

It is easy to see that this universe is plagued by the events horizon.

The case $\delta \geq 1$ is a more interesting one. If $\delta << 1$, then solution will describe the universe, starting from an initial singularity ($\theta = 0$, $t = 0$) and ending up in the final singularity at $t_f = \Lambda^{-1/2}\mathrm{arccoth}\delta$. One shall note that, if $\delta \geq 1+\sqrt{2}$ the strong energy condition will always be satisfied (in fact, universe will end up in singularity long before the time t_v).

When $\delta = 1$ the resulting solution might be denoted as "quasisingular" because $\lim\limits_{t\to\infty} a \sim \lim\limits_{\theta\to\infty} \sinh^{1/2}\theta \ln\coth\frac{\theta}{2} \sim \exp(-\theta/2).$ From this relation one can see that scale factor tends to singularity but never achieves it.

Both singular and quasisingular models contains no events horizons. To show this for the quasisingular case let us integrate the equation for future directed radial null geodesics ($ds^2 = 0$) just like it has been done in the previous subsection: $\Delta\chi \sim \int\limits_0^{\infty} \frac{d\theta}{\sinh^{1/2}\theta \ln\coth\frac{\theta}{2}}.$ This integral diverges because subintegral expression has an exponential asymptot at large θ. Therefore radial null geodesics circumnavigate the universe an infinite number of times as $t \to \infty$. This fact and the homogeneity+isotropy result in the conclusion similar to the one from the previous subsection, namely that such universe possess no events horizon. Absence of events horizon for singularity model can be proved by analogy.

In conclusion let us analyze the equation of state for the generalization of a lambda-radiation model. One can show that the value $w = p/\rho$ is equal to

$$w = -\frac{1}{3} + \frac{2}{3} \frac{(1-\sinh^2\theta)\ln^2 \frac{\coth\frac{\theta}{2}}{\delta}}{(\cosh\theta \ln\frac{\coth\frac{\theta}{2}}{\delta} - 2)^2}. \tag{26}$$

From this relation it follows immediately that for both open and quasisingular models $w \to -1$ at large t (large θ). For the closed model $w = -1/3$ whenever we approach the final singularity. In the initial singularity $w = 1/3$ for all models. Close examination of equation (26) shows that w is always greater than -1 for all cases, i.e. weak energy condition will always be satisfied.

5. Conclusion

We have discussed a simple (and easily automatizable) method of construction of exact solutions of Friedmann equations. Despite simplicity, the method allows for acquirement of solutions characterized by extremely interesting properties. What is more, it appears that the very abundance of the set of solutions that are to be obtained this way leads us to a stunning conclusion: no matter how accurate our astronomical observations are, there exist not just one, but a whole set of solutions that will satisfy the observational data while leading to essentially different dynamics in future. This sudden twist leads us to seemingly unavoidable conclusion about the principle indefiniteness of the future, hidden in the Fridmann equations. For a first glance such conclusion looks really disappointing, rendering useless all our efforts to build a suitable cosmological model describing our universe.

However, everything above-said doesn't mean the impossibility to determine the actual dynamics of the universe in principle. Even though the usual observational methods might not give us the final answer, we can use for example the self-consistensy condition for selection of cosmological models.

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