Non-positive Dimension Spaces

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Abstract: The concept of non-positive dimension spaces is offered. The metric of these spaces is introduced, which bases on the two-sides divergent series theory. It is shown that corollaries of this idea are consistent with solutions of usual equations which have formal analytic continuation to non-positive dimension spaces. In frames of offered theory the point is a dual object what has properties as an usual point so as a stretched space. Physical effects what may be explained with new concept are considered, including the big bang hypothesis.

1. The attempts of a human mind to penetrate into more and more deep "beds" of the creation are inevitable connected with numerous speculative constructions basing on scant experimental facts and their ambiguous interpretation. All these attempts have the same feature as the using of "an analytic continuation" all sum of knowledge which existed hitherto.

In presented work a new mathematical object is introduced, the stretched space with non-positive (whole) number of dimensions, what we denote as *an antispace* briefly. Besides, elements of a new mathematical theory of this object is interesting with point of view a possibility of realization on sufficiently small distances, because of paradoxical properties of an antispace have analogs in physics of a micro world. In addition, the antispace concept permit to offer one possibility of the big bang phenomenon explanation.

2. The adequate mathematical instrument for describing of zero- and minus-spaces *the two-sides divergent series* (TSDS) theory turns out. (Which is the extension of *a one-side divergent series* (DS) concept.)

The interest for using DS has appeared with a mathematical analysis developments, when it have been turned out that DS are useful because of formal operations with them lead to correct results often, which then may be verified with an independent way. In this connection it appears a natural question, what "sum" is necessary to prescribe for a divergent series. This question was developed in detail in works of Euler, Chesaro, Lagrange and other famous mathematicians. The most complete survey see in book [1].

For example, we can ascribe a sense to the expression

$$S = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{1}$$

at all $x \neq 1$. In particular, if $x = 2$ we have $S = 1 + 2 + 4 + ... = -1$. At this example one can see that in frames of DS theory *"a sum" of positive quantities may be a negative one.*

By differentiation (1) with respect to x and putting $x = -1$, we go to the result $S = 1 - 2 + 3 - 4 + ... = 1/4$, which shows that *an alternating DS of whole numbers may have a fraction value of its "sum".*

There are many other paradoxes which have been studied good enough to avoid mistakes in final results. Especially important for presented theory of antispace is the fact that for DS has importance not only a term magnitude but its "position".

For example, we have $1 - 1 + 1 - 1 + 1 - ... = 1/2$, but $1 + 0 - 1 + 1 - 0 - 1 + ... = 2/3$ (see [1]).

The properties of TSDS are still more exotic. In particular, a "sum" of the series

$$S = \sum_{k=-\infty}^{\infty} x^k = 0 \tag{2}$$

is equal to zero at all values of x in virtue of the next formal operations:

$$S = \left(... + \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} \right) + \left(1 + x + x^2 ...\right) = \left(\frac{1}{x} \left(\frac{1}{1 - (1/x)} \right) \right) + (1 - x) = 0 \ .$$

The fact that a great many TSDS have “a sum” which equals zero permits to input a concept of a sum for zero and negative number of terms. It is clear that if we know an analytic expression of a finite sum, for example

$$S_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \ , \tag{3}$$

the formal finding of S_n for any $n \in (-\infty, \infty)$ has not difficulties. In particular, using (3) we find $S_0 = 0, S_{-1} = 0, S_{-2} = -1, S_{-3} = -5, ...$ etc. But how to calculate these values if an explicit expression (3) is unknown?

Let us write TSDS which generalizes (3):

$$S = \sum_{k=-\infty}^{\infty} k^2 = ... + 2^2 + 1^2 + 0^2 + 1^2 + 2^2 + ... = 0 \, . \tag{4}$$

We postulate that

$$S_0 = \sum_{k=1}^{0} k^2 = \sum_{k=-\infty}^{\infty} k^2 = S = 0 \, . \tag{5}$$

Rejecting in (4) the term $k = 0$ we obtain $S_{-1} = 0 - 0^2 = 0$. Rejecting two terms $k = 0, k = -1$ we find $S_{-2} = 0 - 0^2 - 1^2 = -1$. The same way one can get $S_{-3} = 0 - 0^2 - 1^2 - 2^2 = -5$.

Found values S_i coincide with these ones obtained above by formal using of expression (3). So, to find a sum of a negative number TSDS terms it is sufficient to take into account the sum of considering terms with a negative sign.

General formulae using while the working with TSDS may be written in the following way. If a_k is "an analytical function" of number k, we have

$$\begin{aligned} &... + a_{-3} + a_{-2} + a_{-1} + a_0 + a_1 + a_2 + ... = 0 \\ &... + a_{-3} + a_{-2} + a_{-1} + 0 + a_1 + a_2 + ... = -a_0 \\ &... + a_{-3} + a_{-2} + 0 + 0 + a_1 + a_2 + ... = -a_{-1} - a_0 \\ &.. \end{aligned} \tag{6}$$

For following brief presentation will be used the partial cases of expressions (6):

$$\begin{aligned} &... + 1 + 1 + 1 + 1 + 1 + ... = 0 \\ &... + 1 + 1 + 0 + 1 + 1 + ... = -1 \\ &... + 1 + 0 + 0 + 1 + 1 + ... = -2 \\ &.. \end{aligned} \tag{7}$$

The series of this kind may be summed up and multiplied by any number.

3. For construction a metric for spaces with non-positive number of dimensions we start from the known expression for a square of distance between two points in usual n-dimension space:

$$l_{1,2}^2 = \sum_{k=1}^{n} \left[(x_k)_1 - (x_k)_2 \right]^2 . \tag{8}$$

Here $(x_k)_{1,2}$ are Cartesian coordinates of point 1 and 2, respectively.

According to the conception of a sum for zero terms number (5), it is naturally to admit that for zero-dimension space (we do not yet know what is it) take place the metric (8) at $n = 0$:

$$l_{1,2}^2 = \sum_{k=-\infty}^{\infty} \left[(x_k)_1 - (x_k)_2 \right]^2 . \tag{9}$$

For (-1)-dimension space (in according to p.2) it is necessary to ″put out″ from (9) one of coordinates with ″preservation its place″:

$$l_{1,2}^2 = \sum_{k=-\infty}^{-1} \left[(x_k)_1 - (x_k)_2 \right]^2 + 0 + \sum_{k=1}^{\infty} \left[(x_k)_1 - (x_k)_2 \right]^2 . \tag{10}$$

Analogously, ″putting out″ two coordinates, we obtain the metric of (-2)-dimension space, etc.

Differential Laplace operator, which in usual n-dimension space has form

$$\Delta_n = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} , \tag{11}$$

for 0-dimension space is written by the next way (see(5))

$$\Delta_0 = \sum_{k=-\infty}^{\infty} \frac{\partial^2}{\partial x_k^2} \tag{12}$$

and for (-1)-dimension space as

$$\Delta_{-1} = \sum_{k=-\infty}^{-1} \frac{\partial^2}{\partial x_k^2} + 0 + \sum_{k=1}^{\infty} \frac{\partial^2}{\partial x_k^2} . \tag{13}$$

etc.

It turns out that introduced definitions (9)-(12) are consistent with the known formulae for usual n-dimension space, which are extrapolated in a domain of non-positive values of n.

In particular, Laplace equation in n-space for a spherical symmetric case has form

$$\Delta_n \Psi = 0, \quad \Delta_n = r^{1-n} \frac{d}{dr} r^{n-1} \frac{d}{dr}; \quad \Psi = \Psi(r), \quad r \in (0, \infty) . \tag{14}$$

Nontrivial solutions of equation (14) exist for all n:

$$\Psi = r^{2-n}, \; n \in (-\infty, 2), (2, \infty);$$
$$\Psi = \ln r, \; n = 2 . \tag{15}$$

Let us check formulae (15) using Cartesian coordinates at case $n \le 0$. According to (12), for zero-dimensional space one can write

$$\left(\ldots + \frac{\partial^2}{\partial x_{-2}^2} + \frac{\partial^2}{\partial x_{-1}^2} + \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots \right)\Psi = 0 . \tag{16}$$

In virtue of (15) and (9), equation (16) at $n = 0$ is satisfied with function

$$\Psi = r^2 = \ldots + x_{-2}^2 + x_{-1}^2 + x_0^2 + x_1^2 + x_2^2 + \ldots \tag{17}$$

Putting (17) in (16), one can check that series (17) is solution (14) because of (7):

$$\Delta_0 \Psi = \ldots + 2 + 2 + 2 + 2 + 2 + \ldots = 2(\ldots + 1 + 1 + 1 + 1 + 1 + \ldots) = 0 .$$

According to (15) for $n = -1$ we must have

$$\Psi = r^3 = \left(\ldots + x_{-2}^2 + x_{-1}^2 + x_0^2 + x_1^2 + x_2^2 + \ldots\right)^{3/2} . \tag{18}$$

Since

$$\frac{\partial \Psi}{\partial x_k} = 3 r x_k , \quad \frac{\partial^2 \Psi}{\partial x_k^2} = \frac{3}{r} x_k^2 + 3r , \tag{19}$$

after substitution (19) to (13), we find

$$\Delta_{-1} \Psi = \frac{3}{r}\left(\ldots + x_{-2}^2 + x_{-1}^2 + 0 + x_1^2 + x_2^2 + \ldots\right) + 3(\ldots + r + r + 0 + r + r + \ldots) .$$

By (10), the first group of summands equals $(3/r) r^2 = 3r$, and the second one gives $-3r$ in according with (7). So, the total sum is zero. Hence, function (18) in fact satisfies to Laplace's equation at $n = -1$.

The same way the correctness of (15) is proved at any negative n.

Similar checking was executed for Helmholtz equation

$$\left(\Delta_n - a^2\right)\Psi = 0, \quad a^2 = const > 0 ,$$

the equation with variable coefficient

$$\left(\Delta_n - a^2 / r^2\right)\Psi = 0, \quad a^2 = const > 0 ,$$

and the nonlinear equation

$$\Delta_n \Psi - r^{-\nu - 2} \Psi = 0, \quad \nu = \frac{2 - n}{2} \pm \sqrt{\left(\frac{2 - n}{2}\right)^2 + 1} .$$

Hence, the metric (9)-(10) introduced above turns out to be non-contradictory with respect to mathematical formalism of differential equations.

4. Naturally, it is interesting to examine the simplest regular figures in antispace.
A volume of usual n-dimensional cube equals to n-power of its rib length:

$$V_n = h.h....h.h = h^n \quad (n \ times) \quad . \tag{20}$$

For zero-space, taking zero times the product of a rib length, we have

$$V_0 = ...h.h.h.h... = 1 \quad . \tag{21}$$

(The operation with two-side infinity products reduce itself to procedures established for TSDS after taking a logarithm. Therefore, (21) may be obtain from (7).)

To find ″a volume″ of (-1)-dimension cube it is necessary to change one of multipliers (21) by unit:

$$V_{-1} = ...h.h.1.h.h... = h^{-1} \tag{22}$$

One can see that dependence (20) turns out correct for space of any number of dimensions $n \in (-\infty, \infty)$.

Formulae (20) and (21), in spite of their simplicity, give rich in content information about ″geometric″ properties of zero- and minus spaces.

1. In right side of expression (21) we have a dimensionless constant which equals to unit. It is hardly to represent in mind that the product of infinite number of multipliers, having dimension of length, has not dimension at all.

We could avoid this paradox if we assume that ″natural″ length l_f (apparently fundamental) exists, what scale must be used for distances measurement.

2. According to (20) and (22), the more a linear size of a minus-cube the less is its volume. This means that the smaller on volume ″body″ some way may be composes from ″bodies″ of larger size.

3. A ″volume″ of zero-dimension cube does not depend of its size and always equals to an unit.

The same examination was executed for other simplest figures (a sphere, a tetrahedron). Besides ″a volume″ was investigated ″a surface area″, a length of diagonals and was described a row of paradoxes which do not contradict to the offered concept of antispace.

It is of interest, that ″a volume″ of all regular figures in zero-space equals an unit and ″a surface area″ is zero.

The forcing feature of our describing is the circumstance that we represent the objects of less dimension (″antiobjects″) as contained with objects of larger dimensions (points, lines). It may be true this fact merely reflects the bound possibilities of a human mentality.

5. Suppose that the quantum theory formalism is not connected with the antispace conception. This statement is quite arbitrary, but we know antispace properties about so few that we use the slightly possibility its study, among them by formal extrapolation of usual equations to domain $n \le 0$.

It is considered Shrodinger equation

$$\frac{d^2\Psi}{dr^2} - \frac{1-n}{r}\frac{d\Psi}{dr} + \left(\varepsilon - \beta r^{2-n}\right)\Psi = 0, \quad \Psi = \Psi(r), \quad r \in [0,\infty); \tag{23}$$
$$\Psi(0) < \infty, \quad \Psi(\infty) < \infty,$$
$$\varepsilon = const > 0, \quad \beta = const > 0,$$

describing spherically symmetric states of two-particle system.

Magnitude β includes universal constants, particles mass, charge and interaction constants of whatever field. Coefficient ε is proportional to eigenvalue of energy. Laplace’s operator is given with expression (14) and potential energy dependence on distance with solution (15).

It was shown [2] that suitable solutions of (23) exist at $n = 1, 2, 3$ and absent at $n \ge 4$. Hence, a hydrogen-similar matter can not exist in spaces of a higher number of dimensions.

We had executed the same research for cases $n \le 0$. It was established that a correct solutions, having a discrete energetic spectrum, exist at any $n \in (-\infty, 0]$.

The case $n = 0$ is remarkable because of all energetic levels dispose at the same distance between them, so $\varepsilon_m = 2\sqrt{\beta}\,(m+2),\; m = 0, 2, 4, \ldots$. Therefore, we may think that if the quantum mechanical systems in zero-space are exist in fact, they differ by an increased stability.

We note that at $n \le 0$ it turns out $\Psi(0) = 0$.

6. In the capacity of ″a working hypothesis″ of antispace is regarded the next ″figurative″ picture. According to metric (9), zero-dimension space (as a special case of antispace) represents the infinite stretched space with infinite (denumerable) amount of mutually perpendicular axes, where can not be ″select the first axis″.

On the other hand, an usual point (as a special case of usual space) is zero-dimension space too.

To make consistent these two (at first sight mutually exclusive) conceptions it is necessary to admit that the next physical hypothesis takes place .

Hypothesis. *Depending on the interaction method with some (macroscopic or microscopic) object, the zero-object can exhibit its properties or as ″a stretched substance″ or as ″a point″.*

Minus-spaces are obtained by ″rejection″ from zero-space some number of coordinates with obligatory ″preservation a place″ for them.

Here take place a certain similarity with Dirac's hole conception. Nevertheless, this analogy is too rough. The change $n \to -n$ in most cases leads to modification of formulae in principle.

The row of qualitative regularities inherent to zero- and minus- mathematical objects has analogies in a microworld physics.

1. The dualism a wave - a particle calls to mind a dualism of a point, which is simultaneously an infinite stretched space (see Hypothesis).

2. The absence of an energy loss for ″an orbital moving″ of electrons in atom may say that a ″rotation″ take place in antispace, where the radiation laws are unknown.

3. The divergences in quantum field theory have similarity with "non-observed" sums type (6).

4. The more bound is a system state the more is its mass. Another words, at increase of mass ″the size″ of an object reduces (see (22)).

5. In quark systems the interaction force does not reduce with increasing of distance. According to model (22) the last has place at $n \le 1$.

6. Quarks do not come close anyhow. The problem (23) solution indicates that a wave function of two-particles system tends to zero at $r \to 0$ as $\Psi \sim r^{2-n}$.

7. As regards an electron quarks have a fractional electric charge. This phenomenon admits some explanations in frames of DS theory (see p.2).

8. Big bang, the extraction of the Universe from a single point, one can consider as the manifestation of a zero-space dualism. Zero-space had extracted a three-dimension space.

In the large we can admit that beginning with length small enough not only quantum but ″minus-geometric″ properties of a matter become essentially. These properties are not observed in macrocosm in virtue of the inverse dependence of ″a volume″ on ″a size″ (see (20) and (22)).

However, our consideration is not sufficient, of course, for a categorical statement about a real existence of antispaces, although the mathematical theory of zero- and minus-spaces seems to us rich in content.

The more details there are in the book [3].

References

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