Relativistic Discs in Black-Hole Spacetimes with Cosmological Constant

© P. Slany [1,2] and Z. Stuchlik [1,3]

[1] Institute of Physics, Silesian University in Opava, Czech Republic
[2] Email: petr.slany@fpf.slu.cz
[3] Email: zdenek.stuchlik@fpf.slu.cz

Abstract: Basic properties of geometrically thin and thick discs in the Schwarzschild–de Sitter and Kerr–de Sitter backgrounds are summarized. The thin discs are represented by the Keplerian motion of test particles along stable circular orbits in the equatorial plane of given spacetime. The repulsive cosmological constant ($\Lambda > 0$) puts an upper limit on the existence of stable equatorial circular geodesics, as well as reduces the values of constants of motion connected with the stationarity and axial symmetry of the spacetime. The thick discs correspond to toroidal equilibrium configurations of a barotropic test perfect fluid, which are characterized by toroidal equipotential surfaces of the "gravitocentrifugal" potential. To demonstrate the phenomenon we use the simplest case of marginally stable tori orbiting the KdS black hole with uniform distribution of the specific angular momentum, $\ell(r,\theta) = \text{const.}$ Resulting structure of the equipotential surfaces is similar to the well-known Roche lobe in a close binary system. In addition to the critical, i.e. marginally closed, equipotential surface self-crossing in the inner cusp the repulsive cosmological constant enables existence of the critical equipotential surface with the cusp at the outer edge of the torus, enabling outflow of a matter from the torus to the outer space due to the same "Roche lobe overflow" mechanism which enables also an accretion onto the black hole through the inner cusp. Moreover, in the case of very rapidly spinning KdS black holes the rotational velocity of particles (fluid) on stable circular orbits, measured in locally non-rotating frames, is an increasing function of the radius in a small region inside the ergosphere. For the current value of the cosmological constant, $\Lambda = \Lambda_0$, the maximal extensions of thin and thick non-gravitating discs around supermassive black holes (10^6-10^9) M_{sun} are (10-10^2) kpc.

1. Introduction

Various cosmological observations indicate the current value of the vacuum/dark energy density

$$\rho_{vac(0)} \approx 0.73\rho_{crit(0)} \tag{1}$$

where the present value of the critical energy density $\rho_{crit(0)}$ is related with the Hubble parameter H_0 by[1]

$$\rho_{crit(0)} = \frac{3H_0^2}{8\pi}, \; H_0 = 100h \,\text{km s}^{-1}\,\text{Mpc}^{-1}. \tag{2}$$

Putting the dimensionless parameter $h \approx 0.7$, we obtain the relic repulsive cosmological constant to be

$$\Lambda_0 = 8\pi\rho_{vac(0)} \approx 1.3\times10^{-56}\,\text{cm}^{-2}. \tag{3}$$

It is well known that the repulsive cosmological constant, $\Lambda > 0$, changes the expansion rate of the Universe, leading finally to exponentially accelerated stage. The repulsive cosmological constant, however, can play an important role also for a formation and evolution of disk-like structures around supermassive black holes.

Basically we can distinguish two types of discs, depending on the relevance of pressure gradients inside the disc. If the pressure gradients (connected with both gas and radiation) are negligible, the disc is geometrically thin and is well represented by the geodesic motion of test particles around stable circular orbits in the equatorial plane of a given black-hole background. On the other hand, if the pressure gradients are relevant (as it is in a hot, dense and opticaly thick material), the disc is geometrically thick and can be characterized by toroidal equilibrium configurations of barotropic perfect fluid orbiting with prescribed (non-Keplerian) distribution of the specific angular momentum. The simplest is the case of marginally stable tori with constant specific angular momentum distribution [1].

The presence of $\Lambda > 0$ substantially changes the asymptotic structure of black-hole spacetimes, as these become asymptotically de Sitter, not flat spacetimes. Influence of the cosmological constant on stationary disc configurations was analyzed in the framework of the Schwarzschild–de Sitter (SdS) and Kerr–de Sitter (KdS) spacetimes, where the equatorial circular motion of test particles (Keplerian motion) and the

[1] Geometric units $c = G = 1$ are used hereafter.

orbital motion of barotropic test perfect fluid characterized by constant distribution of the specific angular momentum were studied [2–6].

Local kinematics of the disc can be described by a rotational velocity field related to the locally non-rotating frames (LNRF), see [13] for their definition in the Kerr spacetime. Typically the topology of equivelocity surfaces is cylindrical, corresponding to monotonic increase of the velocity with decreasing radius. Quite interestingly, in the case of near-extreme Kerr black holes the rotational velocity of both particles and fluid reveals a hump in its radial profile in the equatorial plane, i.e., there are two radii inside the ergosphere where the radial gradient of the rotational velocity changes its sign, and is positive between these radii [7, 8]. Moreover, the topology of equivelocity surfaces also changes, becoming toroidal. Humpy profiles of the rotational velocity in KdS backgrounds were analyzed in [9, 10, 11].

2. Kerr–de Sitter black-hole spacetime

Geometry of the Kerr–de Sitter spacetime in the Boyer-Lindquist coordinates (t, r, θ, φ) is described by the line element

$$ds^2 = -\frac{\Delta_r}{I^2\rho^2}(dt - a\sin^2\theta\ d\varphi)^2 + \frac{\Delta_\theta \sin^2\theta}{I^2\rho^2}[a dt - (r^2 + a^2)d\varphi]^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \tag{4}$$

where

$$\Delta_r = r^2 - 2Mr + a^2 - \frac{1}{3}\Lambda r^2\left(r^2 + a^2\right), \quad \Delta_\theta = 1 + \frac{1}{3}\Lambda a^2\cos^2\theta,$$

$$I = 1 + \frac{1}{3}\Lambda a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta.$$

The spacetime is characterized by three parameters: central point mass M, rotational parameter or spin a (corresponding to the specific angular momentum of the center), and the positive cosmological constant $\Lambda > 0$. It is convenient to introduce a dimensionless *cosmological parameter*

$$y = \Lambda M^2/3$$
(5)

and put $M = 1$ to get completely dimensionless formulae, hereafter.

Event horizons of the spacetime are given by the condition $\Delta_r = 0$. In the KdS spacetimes a *cosmological horizon* r_C always exists, behind which the geometry is dynamic. The KdS black hole is characterized, in general, by two horizons, the outer horizon and the inner horizon; between these horizons the spacetime is not stationary. When both black-hole horizons coincide, we speak about the *extreme* KdS black hole. It should be stressed that a spin of the extreme KdS black hole depends on the cosmological parameter y (cosmological constant Л) and for Л > 0, $a_{ex} > 1$. Maximal value of the cosmological parameter allowing the KdS black holes is $y_{c(KdS)}$ ✞ 0.05924; corresponding spin of the extreme black hole is $a_{ex(max)}$ ✞ 1.10092 [4].

When the rotational parameter $a = 0$, the KdS spacetime reduces to the Schwarzschild–de Sitter spacetime. The limiting value of the cosmological parameter enabling the SdS black holes is $y_{c(SdS)} = 1/27$; for $y > 1/27$ there are no event horizons, no stationary regions, and the spacetime is dynamic everywhere.

2.1 Equatorial circular orbits of test particles

Free test particles follow geodesic curves in the spacetime. The geodesic equations in a separated and integrated form were first obtained by Carter [12]. In the KdS spacetime, Carter's equations for geodesics in the equatorial plane ($\theta = \pi/2$) have the form [4]

$$r^2 \frac{dr}{d\lambda} = \pm\sqrt{R(r)}, \tag{6a}$$

$$\begin{aligned} r^2 \frac{d\varphi}{d\lambda} &= -IP_\theta + \frac{aIP_r}{\Delta_r}, \\ r^2 \frac{dt}{d\lambda} &= -aIP_\theta + \frac{(r^2+a^2)IP_r}{\Delta_r} \end{aligned} \tag{6b}$$

where

$$R(r) = P_r^2 - \Delta_r\left(m^2r^2 + K\right), \quad P_r = I\tilde{E}\left(r^2+a^2\right) - Ia\Phi, \quad P_\theta = I\left(a\tilde{E} - \Phi\right), \quad K = I^2\left(a\tilde{E}-\Phi\right).$$

The proper time ф of a particle with the rest mass *m* is related to the affine parameter *л* as *ф* = *m л*. Note that the constants of motion, $\tilde{E}$ and Ц, connected with the stationarity and axial symmetry of the spacetime, respectively, cannot be interpreted as the energy and axial angular momentum at radial infinity, since the spacetime is not asymptotically flat. Further we define the specific energy and the specific angular momentum by the relations

$$E \equiv \frac{\tilde{E}}{Im}, \quad L \equiv \frac{\Phi}{Im}. \tag{7}$$

Analyzing Carter's equation for the radial motion (6a) it was shown that the equatorial motion of a test particle with rest mass *m* is governed by the *effective potential* [4]

$$\Phi_{eff} = \frac{1}{r^2}\left[aX + \sqrt{\Delta_r\left(r^2+X^2\right)}\right], \tag{8}$$

which is parametrized by the axial parameter $X = L - aE$. In stationary regions of the spacetime ($\Delta_r \geq 0$) the motion of a test particle with the specific energy *E* is restricted by the condition $E \geq \Phi_{eff}$.

Equatorial circular timelike geodesics are given by local extrema of Ц$_{\mathrm{eff}}$. There are either one local maximum or two local maxima and one local minimum in between them. Local maxima correspond to unstable orbits, while local minimum determines stable (with respect to any radial perturbation) circular orbit of a test particle. Existence of the stable circular orbit depends on spacetime parameters (*y, a*) and a concrete value of the axial parameter *X*.. If the stable circular orbit exists, two unstable circular orbits are also present. In KdS (SdS) spacetimes there are two marginally stable orbits, the *inner marginally stable orbit*, corresponding to a value of *X* for which the inner local maximum and the local minimum coincide, and the *outer marginally stable orbit*, corresponding to *X* for which the outer local maximum and the local minimum coincide. If both local maxima are of the same high, they correspond to the *inner* and *outer marginally bound orbit*. Otherwise one unstable circular orbit always exists.

The specific energy and specific angular momentum of a particle on the equatorial circular orbit are given by the relations (see [4] for their alternative but equivalent formulation)

$$E_\pm(r;a,y) = \frac{r^{3/2} - 2r^{1/2} - r^{3/2}(r^2+a^2)y \pm a(1-yr^3)^{1/2}}{r^{3/4}\left[r^{3/2}(1-a^2y) - 3r^{1/2} \pm 2a(1-yr^3)^{1/2}\right]^{1/2}},$$

$$L_\pm(r;a,y) = \pm\frac{(r^2+a^2)(1-yr^3)^{1/2} \mp \left[2ar^{1/2} + ar^{3/2}(r^2+a^2)y\right]}{r^{3/4}\left[r^{3/2}(1-a^2y) - 3r^{1/2} \pm 2a(1-yr^3)^{1/2}\right]^{1/2}}.$$

(9)

There are two families of orbits: a plus-family and a minus-family, corresponding to upper and lower signs, respectively, in relations (9). Clearly, there are two reality conditions on the existence of circular geodesics:

$$r \leq r_s \equiv y^{-1/3} \tag{10}$$

and

$$r^{3/2}(1-a^2y)-3r^{1/2} \pm 2a\,(1-yr^3)^{1/2} > 0\,. \tag{11}$$

The first condition (10) defines the *static radius*, i.e. the orbit where a freely falling observer with only time-component U^t of its 4-velocity being non-zero can reside. This orbit is, however, unstable against radial perturbations and counterrotating (in terms explained below). Second condition (11) is connected with the fact that the orbits, for which the left-hand side of the relation (11) is zero, correspond to null circular geodesics, i.e. the orbits of photons.

Analogical relations for the motion in SdS spacetimes can be obtained from relations (9) – (11) by putting $a = 0$. Note that a location of the static radius is independent of the rotational parameter a. Photon circular orbit in SdS spacetimes is located at the radius $r = 3$, independently of the cosmological parameter y. Detailed discussion of the test-particle motion in the S(a)dS spacetime is presented in [2].

Any stationary disk configurations can exist only in those KdS (SdS) spacetimes in which a motion along stable circular geodesics is possible. Stable plus-family equatorial circular orbits can exist only in those KdS spacetimes for which $y \le y_{stab}^{(+)}$ ✞ 0.069, while the stable minus-family equatorial circular orbits (and stable equatorial circular orbits in SdS spacetimes) can exist only in the spacetimes with $y \le y_{stab}^{(-)} = 12/15^4$ ✞ 0.00024, see [2, 4].

In asymptotically flat spacetimes the direction of motion along a circular orbit can be determined from the point of view of a static observer at infinity or, equivalently, from the point of view of a locally non-rotating frame (LNRF). In the KdS spacetime we can use only the method based on the point of view of the locally non-rotating frames. LNRF in the KdS spacetime is defined by the basis tetrad of 1-forms [4]

$$\begin{aligned}
e^{(t)} &= \left(\frac{\Delta_r \Delta_\theta \rho^2}{I^2 A}\right)^{1/2} dt, \quad e^{(\varphi)} = \left(\frac{A\sin^2\theta}{I^2\rho^2}\right)^{1/2} \left(d\varphi - \omega\, dt\right) \\
e^{(r)} &= \left(\frac{\rho^2}{\Delta_r}\right)^{1/2} dr, \quad e^{(\theta)} = \left(\frac{\rho^2}{\Delta_\theta}\right)^{1/2} d\theta,
\end{aligned} \tag{12}$$

where

$$A = \left(r^2 + a^2\right)^2 \Delta_\theta - a^2 \Delta_r \sin^2\theta \tag{13}$$

and the angular velocity of the LNRF $\omega = d\varphi / dt = -g_{t\varphi} / g_{\varphi\varphi}$ is given by the relation

$$\omega = \frac{a}{A}\left[\left(r^2 + a^2\right)\Delta_\theta - \Delta_r\right]. \tag{14}$$

The orbit is corotating (counterrotating), if locally measured azimuthal component of particle's 4-momentum $P^{(\varphi)} = P^\mu e_\mu^{(\varphi)}$ is positive (negative). It is shown in [4] that the sign of $P^{(\varphi)}$ is determined by the sign of angular momentum L.

Combined analysis of circular orbits with respect to the orientation and stability reveals that minus-family orbits (both stable and unstable) are always counterrotating. Plus-family orbits are usually corotating but counterrotating plus-family orbits also exist. In all (black-hole and naked-singularity) KdS backgrounds, the plus-family orbits become counterrotating in the vicinity of the upper limit for their existence (the static radius or the counterrotating photon circular orbit); these orbits are, however, unstable. In naked-singularity backgrounds with the rotational parameter low enough, the counterrotating plus-family circular orbits exist also in the vicinity of the ring singularity; these orbits are both unstable (located under the inner marginally stable orbit) and stable. For sufficiently high values of the cosmological parameter, $y \approx 0.06$ (thus only naked-singularity spacetimes are possible), all stable plus-family orbits are counterrotating. Moreover, as in the Kerr naked-singularity spacetimes [14], when the rotational parameter a is very close to the extreme-hole state, the counterrotating stable plus-family orbits for which the specific energy is negative, $E_+ < 0$, exist.

2.2 Toroidal configurations of a barotropic perfect fluid

Now we shall consider a perfect fluid described by the stress-energy tensor

$$T^{\mu\nu} = (\varepsilon + p)U^{\mu}U^{\nu} + pg^{\mu\nu}, \tag{15}$$

where *e* and *p* are the proper energy density and isotropic pressure of the fluid, which motion is characterized by a 4-velocity field

$$U^{\mu} = \left(U^{t}(r,\theta), 0, 0, U^{\varphi}(r,\theta)\right), \tag{16}$$

and prescribed distribution of the specific angular momentum[2]

$$\ell(r,\theta) = -\frac{U_{\varphi}}{U_{t}}. \tag{17}$$

The specific angular momentum ℓ is related to the angular velocity $\Omega = U^{\varphi}/U^{t}$ through the relation with metric coefficients

$$\Omega = -\frac{\ell g_{tt} + g_{t\varphi}}{\ell g_{t\varphi} + g_{\varphi\varphi}}. \tag{18}$$

Projecting the covariant energy-momentum conservation law, $\nabla_{\mu}T^{\mu\nu} = 0$, onto the hypersurface orthogonal to the 4-velocity U^{μ}, we obtain the relativistic Euler equation in the axially symmetric form [1, 15]

$$\frac{\partial_k p}{\varepsilon + p} = -\partial_k(\ln U_t) + \frac{\Omega \partial_k \ell}{1 - \Omega\ell} \tag{19}$$

where *k* = *r, u*.

For a barotropic fluid, i.e., for the fluid with the equation of state *p* = *p*(*e*), the surfaces of constant pressure are given by the equipotential surfaces of the potential *W*(*r, u*) defined by the relations [1, 15]

$$\int_0^p \frac{dp}{\varepsilon + p} = \ln(U_t)_{in} - \ln U_t + \int_{\ell_{in}}^{\ell} \frac{\Omega\, d\ell}{1 - \Omega\ell} \equiv W_{in} - W; \tag{20}$$

the subscript “in” refers to the inner edge of the configuration. To obtain the explicit form of the potential *W* we need to define the “rotational law”, i.e. the function *Щ* = *Щ*(ℓ). The equipotential surfaces, given by the condition *W*(*r, u*) = const., can be closed or open. Moreover, there is a special class of *critical surfaces* self-crossing in the cusp(s), which can be either marginally closed or open. The closed–toroidal–equipotential surfaces determine stationary configurations (tori).

Topological properties of equipotential surfaces, in general, seem to be rather independent of the distribution of the specific angular momentum ℓ(*r, u*), see, e.g., [16, 17]. The simplest, however unrealistic, are configurations with constant specific angular momentum,

$$\ell(r, u) = \text{const.} \tag{21}$$

In this special case the potential is given by very simple formula

$$W(r,\theta) = \ln U_t(r,\theta). \tag{22}$$

[2] The specific angular momentum ℓ is the angular momentum per *total* energy instead of *L*, which corresponds to the angular momentum per *rest* energy, $\ell = L/E$.

Orbits where $\partial_k W(r,\theta) = 0$ (*k = r, u*) correspond to free-particle orbits (geodesics), because the pressure-gradient forces are zero there. Moreover, at the center of any torus the pressure attains the extreme value (maximum) and matter must follow a stable geodesic there. Thus, fluid tori can exist only in the spacetimes with stable circular geodesics.

In the KdS spacetime the potential (22) takes the form [5]

$$W(r,\theta) = \ln\left[\frac{\rho^2}{I^2}\frac{\Delta_r \Delta_\theta \sin^2\theta}{(r^2+a^2-a\ell)^2 \Delta_\theta \sin^2\theta - (\ell - a\sin^2\theta)^2 \Delta_r}\right]^{1/2}. \tag{23}$$

The form of the potential in the SdS spacetime can be obtained by performing the limit $a \to 0$ in (23). Detailed discussion of toroidal structures in the SdS black-hole spacetime can be found in [3].

All relevant properties of the equipotential surfaces are revealed by behaviour of the potential in the equatorial plane (*u = p*/2). Reality conditions of the function *W*(*r, u = p*/2) imply that the fluid with a given distribution of the specific angular momentum can occupy the stationary regions in the equatorial plane where

$$\ell_{ph-} < \ell < \ell_{ph+}; \quad \ell_{ph\pm} = a + \frac{r^2}{a \pm \sqrt{\Delta_r}}. \tag{24}$$

$\ell_{ph\pm}$ correspond to the effective potentials governing the equatorial motion of photons; for their alternative formulation see [18]. Local extrema of the function *W*(*r, u = p*/2) lie at those radii where the specific angular momentum coincides with the specific angular momentum of test particles moving on the geodesical (Keplerian) circular orbits, i.e., where

$$\ell = \ell_{K\pm}(r;a,y) \equiv \pm \frac{(r^2+a^2)(1-yr^3)^{1/2} \mp \left[2a\,r^{1/2} + a\,r^{3/2}(r^2+a^2)y\right]}{r^{3/2} - 2r^{1/2} - r^{3/2}(r^2+a^2)y \pm a(1-yr^3)^{1/2}}. \tag{24}$$

Toroidal configurations exist for such distribution of ℓ(*r, u*) in the disc which intersects the Keplerian distribution of the specific angular momentum in the part(s) corresponding to stable circular orbits. In black-hole backgrounds, stationary toroidal configurations exist for $\ell \in (\ell_{ms(i)}, \ell_{ms(o)})$, where $\ell_{ms(i)}$ ($\ell_{ms(o)}$) corresponds to the Keplerian specific angular momentum on the inner (outer) marginally stable orbit. The same is true also in most of the naked-singularity backgrounds, however, exceptions exist, concerning the plus-family discs in naked-singularity backgrounds with the rotational parameter low enough to admit counterrotating stable plus-family circular geodesics. In fact, there are naked singularities around which the stationary tori can exist for any value of ℓ [5].

2.3 Rotational velocity profiles

Rotational velocity $V^{(\varphi)}$ corresponds to locally measured azimuthal component of the 3-velocity in the LNRF. In the KdS spacetime the LNRF-tetrad has the form (12) and the rotational velocity is given by the relation

$$V^{(\varphi)} = \frac{U^\mu e_\mu^{(\varphi)}}{U^\nu e_\nu^{(t)}} = \frac{A\sin\theta}{\rho^2\sqrt{\Delta_r \Delta_\theta}}(\Omega - \omega), \tag{25}$$

where *Щ* is the angular velocity of a matter determined by the specific angular momentum ℓ through the relation (18), and *щ* is the angular velocity of the LNRF (14).

For Keplerian motion of test particles in the equatorial plane the corresponding Keplerian angular velocity is given by the relation [5]

$$\Omega_{K\pm} = \frac{1}{a \pm \sqrt{r^3/(1-yr^3)}}, \tag{26}$$

and the rotational velocity of test particles is therefore described by the relation [10]

$$V_{K\pm}^{(\varphi)}(r;a,y) = \frac{(r^2+a^2)\sqrt{1-yr^3} \mp a\sqrt{r}\left[2+r(r^2+a^2)y\right]}{\sqrt{\Delta_r}\,[a\sqrt{1-yr^3} \pm r\sqrt{r}]}. \tag{27}$$

For the uniform distribution of the specific angular momentum (21) the rotational velocity profiles are described by the function [10]

$$V_{uni}^{(\varphi)}(r,\theta;a,y,\ell) = \frac{\rho^2 \ell \sqrt{\Delta_r \Delta_\theta}}{\{A - [\Delta_\theta(r^2+a^2) - \Delta_r]a\ell\}\sin\theta}. \tag{28}$$

3. Keplerian discs

Radial profile of the angular velocity Щ = dџ/d*t* of a thin, Keplerian disc orbiting the KdS black hole (or even naked singularity) is given by the relation (26). In the most extended Keplerian accretion disc a matter in the disc spirals from the outer edge located at the outer marginally stable orbit, $r_{out} \approx r_{ms(o)}$, through the sequence of stable circular orbits down to the inner edge located at the inner marginally stable orbit, $r_{in} \approx r_{ms(i)}$, losing the energy and angular momentum due to the viscosity. It is easy to show by direct analysis of relations (9) and (26), see [4], that the necessary conditions for such differential rotation, i.e. $d\Omega_{K+}/dr < 0, dE_+/dr \geq 0, dL_+/dr \geq 0$ for plus-family discs, and $d\Omega_{K-}/dr > 0, dE_-/dr \leq 0$, $dL_-/dr \leq 0$ for minus-family discs are fulfilled.

We can define a theoretical efficiency of accretion in the Keplerian disc by the difference of the specific energies of a particle on the outer and the inner marginally stable orbit, $\eta = E_{ms(o)} - E_{ms(i)}$. For Keplerian discs corotating extreme KdS black holes, the accretion efficiency reaches maximum value з ~ 0.43 for *y* = 0 (the extreme Kerr black hole) and tends to zero for $y \to y_{c(KdS)} \approx 0.059$, the maximum value of *y* admitting black holes. For plus-family discs orbiting the KdS naked singularity with the rotational parameter close to the extreme-hole state the accretion efficiency can exceed the efficiency of annihilation processes, з > 1. Again, the maximum is for the Kerr naked-singulariy (*y* = 0), з ~ 1.57 [14], and for $y \to y_{c(KdS)}$ the maximal efficiency tends to zero. Due to strong discontinuities in the properties of plus-family orbits for extreme black holes and naked singularities with $a \to a_{ex}$, a hypothetical conversion of naked singularity into extreme black hole, induced by the accretion in plus-family discs, leads to an abrupt instability of the innermost parts of the discs around the naked singularity that can have strong observational consequences; for more details see [4, 14].

Local kinematics of the Keplerian disc is described by the Keplerian rotational velocity profile (27). In all KdS spacetimes admitting stable minus-family orbits the rotational velocity $V_{K-}^{(\varphi)}$ monotonically decreases with the radius. On the other hand the rotational velocity profile of plus-family orbits, $V_{K+}^{(\varphi)}(r)$, contains a hump near the inner marginally stable orbit for discs orbiting near-extreme black holes or naked singularities. This effect of non-monotonicity was at first discussed by Aschenbach [7] in the case of Kerr black holes. The spin of the Kerr black hole, for which the Aschenbach effect exists, must be sufficiently high, $a \geq 0.9953$. Repulsive cosmological constant (Л > 0) shifts the minimal spin even to higher values while the attractive cosmological constant (Л < 0) lowers the minimal spin [9, 10]. For cosmological parameters from the unterval $-10^{-3} < y < 10^{-3}$ the minimal spin, for which the Aschenbach effect exists, is approximately given by a linear relation [10]

$$a_{\min} \approx 0.99529 + 1.35395\,y. \tag{29}$$

4. Marginally stable barotropic perfect fluid tori

According to general criterium of stability, $\ell(r,\theta) \geq 0$ [19], the tori or their part with uniform distribution of the specific angular momentum (21), are marginally stable. Moreover, they are capable to produce maximal luminosity at all [16]. As it has been already mentioned, toroidal equipotential surfaces determine stationary configurations. The fluid can fill any toroidal equipotential surface; at the surface of the torus pressure vanishes but its gradient remaines non-zero. Existence of the critical–marginally closed–equipotential surface enables an outflow from the torus through the cusp of the equipotential surface (located in the equatorial plane), when the surface of the disc overcomes the critical surface. The outflow, like an accretion onto the black hole, is thus driven by a violation of the hydrostatic equilibrium in the torus, rather than by a viscosity of the fluid [15].

In the KdS (SdS) spacetime three qualitativelly different types of tori can exist [3, 5]. The first type of configurations corresponds to the well known *accretion discs*. The cusp of the critical marginally closed equipotential surface is located at the inner edge of torus between the inner marginally stable and the inner marginally bound circular geodesic, $r_{mb(i)} \leq r_{in} \leq r_{ms(i)}$ of the spacetime. Moreover, there is another critical surface, self-crossing in the outer cusp, which is open. The second type of configurations is so-called *excretion disc*. The cusp of the critical marginally closed equipotential surface is located at the outer edge of the torus between the outer marginally stable and the outer marginally bound circular geodesic, $r_{ms(o)} \leq r_{out} \leq r_{mb(o)}$. In the special case of the so-called *marginally bound accretion disc* both the cusps belong to the same critical equipotential surface, and their locations coincide with the inner and outer marginally bound geodesics, $r_{in} = r_{mb(i),}\ r_{out} = r_{mb(o)}$. The marginally bound accretion disc is, therefore, the most extended torus in a given KdS spacetime.

Existence of the outer cusp in the structure of equipotential surfaces is caused fully by the repulsive cosmological constant (Л > 0). It should be stressed that the outer cusp plays an important role also for the accretion discs. After a large overfilling of the critical surface with the inner cusp, when also the critical surface with the outer cusp is overfilled, the outflow through the outer cusp begins to complement the accretion inflow. As shown by Rezzolla et al. [20] in the case of perfect fluid tori orbiting SdS black holes, the excretion outflow through the outer cusp is able to stabilize accretion discs against the so-called *runaway instability*, discussed, e.g., in [21].

Local kinematical properties of the marginally stable fluid tori are described by the rotational velocity profile (28). In most cases the isovelocity surfaces inside the torus have cylindrical topology; the rotational velocity $V^{(\varphi)}$ decreases with increasing radius. However, as in the case of Keplerian discs orbiting near-extreme KdS black holes, the region of increasing $V^{(\varphi)}$ with increasing radius also exists inside the torus. This effect is accompanied by the topology change of isovelocity surfaces, which become toroidal. These toroidal surfaces are separated from the cylindrical ones by two critical surfaces with the cusp in the equatorial plane, where one of the critical surfaces is marginally closed and the second is open. Nevertheless, there is also a special case of only one critical marginally closed isovelocity surface containing both the cusps. The cusps and the central ring correspond to the local maxima and minimum, respectively, of the rotational velocity in the equatorial plane. The whole region with toroidal isovelocity surfaces is located inside the ergosphere of the black hole. Again, existence of the Aschenbach effect in marginally stable tori was at first analyzed in tori around Kerr black holes [8]. It was found that a minimal spin of the Kerr black hole, necessary for the existence of the effect in the marginally stable torus, is $a_{\min} \approx 0.9998$, and changes with the cosmological parameter $y \in (-10^{-3}, 10^{-3})$ approximately as [11]

$$a_{\min} \approx 0.99986 + 1.00977\, y\,. \tag{30}$$

5. Concluding remarks

Finally, we shall give an idea on scales, at which the cosmological constant substantially influences discs around black holes, by expressing locations of the outer edge of the Keplerian disc and fluid tori for the current value of the cosmological constant Л$_0$ in astronomical units. Recall that the outer edge of the Keplerian disc is located close to the outer marginally stable orbit, while the outer edge of the barotropic perfect-fluid torus can be extended up to the outer marginally bound orbit, which is located very close to the static radius of the spacetime for the current value of the cosmological constant and typical masses of black holes. The results are presented in Table 1. Remarkably, but rather by coincidence, the dimensions of discs around supermassive black holes (10^6-10^9) M_{sun} are comparable with the dimensions of galaxies. It should be stressed, however, that the presented values correspond to non-gravitating discs and tori.

Table 1: Cosmological parameter, and corresponding mass of the central extreme KdS black hole, radius of the outer marginally stable circular geodesic, and the static radius of a given KdS spacetime in astronomical units for the current value of the cosmological constant Л = $Л_0$ ✞ 1.3 x 10^{-56} cm^{-2}. The last line shows the central mass-density of an adiabatic non-relativistic torus (with the adiabatic index ƨ = 7/5) orbiting the KdS black hole with *a/M* = 0.9 and Л = $Л_0$, for which $m_{disc} \approx M_{BH}$.

Y	10^{-44}	10^{-42}	10^{-40}	10^{-34}	10^{-32}	10^{-30}	10^{-28}
M/M_{sun}	10	10^2	10^3	10^6	10^7	10^8	10^9
$r_{ms(o)}$ / kpc	0.15	0.31	0.67	6.7	15	31	67
r_s / kpc	0.23	0.50	1.1	11	23	50	110
c_c / kg m^{-3}	–	–	–	–	10^{-12}	10^{-13}	10^{-14}

Relevance of the "test-disc" approximation can be analyzed for the non-relativistic adiabatic perfect-fluid tori described by the adiabatic equation of state $p = K\rho^{\gamma}$, where $\gamma = 1 + 1/n$ is the adiabatic index and the non-relativistic limit means that $p << \rho \approx \varepsilon$; *c* is the rest-mass density. The total mass of the torus is given by Tolman equation [15]

$$m = \int_{disc} \left(-T_t^t + T_r^r + T_\theta^\theta + T_\varphi^\varphi\right)\sqrt{-g}\, dr\, d\theta\, d\varphi; \quad g = \det(g_{\mu\nu}) \tag{31}$$

leading to the expression [6]

$$m = 2\pi\rho_c \int_{disc} \left[\frac{1 + \ell\,\Omega(r,\theta)}{1 - \ell\,\Omega(r,\theta)}\right]\left[\frac{\exp\{W_{in} - W(r,\theta)\} - 1}{\exp\{W_{in} - W_c\} - 1}\right]^n \left(r^2 + a^2 \cos^2\theta\right)\sin\theta\, dr\, d\theta \tag{32}$$

where ρ_c denotes the mass-density in the center of the torus characterized by the potential value W_c. The central mass-densities ρ_c of the fluid with the adiabatic index ƨ = 7/5 (it corresponds, e.g., to a gas of diatomic molecules), for which the mass of the torus is comparable with the mass of the KdS black hole with the spin *a* = 0.9 and Л = $Л_0$, are also presented in Table 1. The limiting central mass-densities are really small but they are still much higher than, e.g., typical densities of current Giant Molecular Clouds 10^{-18} kg/m^3.

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