Local Effects of the Cosmological Constant

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Abstract: If the positive cosmological constant Λ explains the recently discovered accelerated stage of the universe it is, at least effectively, beside the Newtonian constant $G_N = m_{Pl}^{-2}$, the second fundamental constant of gravity. Besides its cosmological implications, Λ affects also astrophysical properties of large structures. We will probe into these matters by using virial equation, hydrostatic equilibrium and the Schwarzschild-de Sitter metric. It will be shown that the effects can be non-negligible. Moreover, it is legitimate to put forth the question how the mass scale $m_\Lambda = \sqrt{\Lambda} << m_{Pl}$ or combinations with the dimension of mass like $\sqrt{m_{Pl} m_\Lambda}$, $(m_{Pl}/m_\Lambda)^n m_{Pl}$, etc. alter our views not only of large scale astrophysics but also our expectations for quantum gravity. We make use of the generalized uncertainty principle and black body radiation to investigate these issues.

1. Introduction:

The resurrection of the cosmological constant $\Lambda = 8\pi G_N \rho_{VAC}$ in an accelerated universe [1] (or, rather in rescue to explain the stage of the accelerated universe) has awakened old controversies about the true nature of this constant. Above all, the "to be or not to be" of Λ has been an eagerly discussed topic up to now. It is usually based on the contribution to vacuum energy from zero-point energy in quantum field theory. The latter is a divergent integral regularized by a cut-off chosen to be at Planck scale. The result is 10^{125} too large which prompted many physicists to search for a convincing mechanism forcing Λ to become zero. No such, generally accepted mechanism has been found.[1]

Instead, as for now many physicists and cosmologists accept $\Lambda > 0$ as a possible explanation of the accelerated universe (and the related problem of the age globular clusters) which in terms of the one of the Friedmann's equations reads:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p(\rho)) + \frac{\Lambda}{3} > 0 \qquad (1)$$

Note that $\dfrac{\ddot{a}}{a}$ without the inclusion of Λ would be always negative. The evidence for a new cosmology (dark energy as it is called) is gaining speed and comes not only from supernovae type Ia surveys [3], but also from cosmic microwave background [4], baryon acoustic oscillations [5] and weak lensing [6].

Having accepted $\Lambda > 0$ as a possible theory to explain these facts, we can go back to see what we had to modify in order to reach at eq. (1). Although this seems like a mere conceptual undertaking, it has to do with a misunderstanding sometimes encountered and rooted in the name "Cosmological Constant" (in other words, does Λ affect only Cosmology?). It is actually the Einstein tensor $G_{\mu H}$ which got altered to:

$$\hat{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} \qquad (2)$$

and not the energy-momentum tensor $T_{\mu H}$ connected to $\hat{G}_{\mu\nu}$ by Einstein's equations:

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[1]Indeed, arguments can be put forward that the cut-off of Planck scale is wrongly chosen [2] and a different well-motivated value, gives a different result.

$$\hat{G}_{\mu\nu} = \kappa T_{\mu\nu} \qquad (3)$$

This simple fact makes the modification **universal** as opposed to situation-specific which would be the case, had we changed $T_{\mu H}$ for cosmology. This is to say that in spite of the name "**Cosmological** Constant", Λ will not only affect cosmological features, but any local physics which emerges from now modified Einstein's equations. In a way, this seems obvious (albeit not always appreciated) as e.g nobody doubts that with the inclusion of Λ the Schwarzschild metric becomes now the Schwarzschild-de Sitter metric, and the Newtonian limit receives modifications[7]:

$$\nabla^2\phi = 4\pi G_N\rho - \Lambda \qquad (4)$$

The relevant question is rather: are the local effects of Λ important?[2]

2. Local aspects of Λ

If $\Lambda = 8\pi G_N \rho_{VAC}$ is the correct model to account for the accelerated universe, we ought to have $c_{vac} \approx 0,7 c_{crit}$. With this value almost all scales set by Λ are of cosmological order of magnitude[3]. Hence it seems that even if Λ affects, in principle, local physics, it does so in a negligible way. However, it can be immediately seen that such an argument is fallacious as it discards the existence of other local scales which in conjunction with Λ can give rise to an observable of astrophysical relevance. Such an example is the motion in Schwarzschild-de Sitter metric [9].

$$ds^2 = -e^\nu dt^2 + e^{-\nu}dr^2 + r^2 d\theta^2 + r^2\sin^2 d\phi^2 \qquad (5)$$

with

$$e^\nu = 1 - \frac{2r_s}{r} - \frac{r^2}{3r_\Lambda^2} \quad ; \quad r_s = G_N M \quad ; \quad r_\Lambda = \frac{1}{\sqrt{\Lambda}} \qquad (6)$$

The motion of a test-body in such a metric is determined according to the equation:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + U_{eff}(r) = C = Const \qquad (7)$$

with:

$$U_{eff}(r) = -\frac{r_s}{r} + \frac{r_l^2}{2r^2} - \frac{r_s r_l^2}{r^3} - \frac{1}{6}\frac{r^2}{r_\Lambda^2} \qquad (8)$$

playing here the role of an effective potential. Three length scales enter the problem: r_s, r_Λ and r_l (angular momentum per unit mass). The first three terms conspire to give U_{eff} a local maximum close to the center and a local minimum (which is the standard minimum of planetary motion). In contrast, the Newtonian part $-\frac{r_s}{r}$ and $-\frac{1}{6}\frac{r^2}{r_\Lambda^2}$; combine to give U_{eff} a local maximum beyond which U_{eff} is a monotone decreasing function. As a result, all bound orbits can happen only within $r < r_{max}$ where r_{max} is the position of the last local maximum

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[2] Interestingly, this issue has been addressed before and stirred another controversy (See [8] and references therein). The author of [8] summarizes the work of several authors as follows:

"The essential difficulty with a relativistic theory in which Λ is positive is that of accounting for the formation and condensation in terms of gravitational instability; for, to use **force** metaphor, the present expansion indicates that the force of cosmic repulsion exceeds those of gravitational attraction. This is not likely to disturb the stability of systems (such as the galaxy) of high average density, but it is likely to present new condensation in regions of low density".

[3] The mass $m_\Lambda = \sqrt{\Lambda} \approx 3x10^{-42}$ is an exception to this rule.

due to Λ. We can estimate r_{max} by setting $r_l=0$. The calculation gives:

$$r_{max} = \sqrt[3]{r_s r_\Lambda^2} \qquad (9)$$

Since $r_\Lambda \approx H_0^{-1}$ (Hubble radius) is a large number compared to r_s, the resulting r_{max} is of astrophysical order of magnitude and its specific values in dependence of the mass M bear strange coincidences [9]. With M the mass of the black hole in the center of our galaxy ($M \approx 10^6 M_{sun}$), $r_{max} \approx 10 Kpc$, which is roughly the order of magnitude of the extension of the galaxy.

Another example of the impact of Λ on local physics is gravitational equilibrium [10]. There the parameter of importance is the density of the object i.e c_{vac} will play the decisive role. The adequate tool to probe into equilibrium matters is the virial theorem [10].

$$4K_{ij} + 2W_{ij} + \frac{2}{3}\Lambda I_{ij} = 0 \qquad (10)$$

Where K_{ij} is the kinetic part, W_{ij} the potential part and I_{ij} the inertial tensor. Denoting the trace of the tensors by K, W and I and making use of the fact that $K \geq 0$, we obtain the inequality:

$$-\frac{1}{3}\Lambda I + |W| \geq 0 \qquad (11)$$

which is trivial in case of $\Lambda = 0$. To illustrate the relevance of this inequality, let us assume c=const. we obtain [10]:

$$\rho \geq A\rho_{vac} \qquad (12)$$

Where A is a dimensionless ratio of two integrals depending only on the geometry (shape) of the body. As $c_{vac} \approx c_{crit}$ again it seems hopeless that (12) is of any phenomenological importance. Indeed, for spherically symmetric objects A=2. But for non-spherical objects A can be quite sizable (factor 10^2 or 10^3 if the deviation from sphericity is large). In passing we note that the dark matter density around astrophysical objects might be $\sim 200 c_{crit}$.

If the dark matter distribution follows the non-spherical shape of the luminous part (oblate and prolate galaxies and even galaxy clusters are known), then the relevance of (12) is obvious. Other local effects were discussed in [11].

Interestingly, Newtonian hydrostatic equilibrium and its relativistic counterpart via Tolman-Oppenheiner-Volkoff equation, gives for spherical symmetry $\overline{\rho}_b < 2c_{vac}$, where:

$$\overline{\rho}_b = \overline{\rho}(R) = \frac{3}{4\pi}\frac{m(R)}{R^3} \qquad (13)$$

And R is the size of the object (see [9] and the references therein). We can demonstrate the effect of Λ in gravitational equilibrium using a slightly different approach.

In place of the inequality, we can maintain K≠0 and solve the scalar virial equation for the mean velocity of the components in the astrophysical object under consideration. For an ellipsoid of constant density we obtain:

$$<v^2> \propto \left(\frac{3}{4}\frac{\rho}{\rho_{vac}}\Gamma - 1\right) \qquad (14)$$

where Γ for an prolate type is:

$$\Gamma \approx \left(\frac{a_1}{a_2}\right)^3 \ln\left(\frac{1+e}{1-e}\right) \qquad (15)$$

with:

$$e = \sqrt{1 - \frac{a_1^2}{a_2^2}} \quad (16)$$

If for instance, c/c_{crit} is 10^3, it suffices to have $\frac{a_1}{a_2} \approx 10^{-1}$, in order that the mean velocity approach to zero, an effect due to Λ. This is an explicit example of the effect of Λ combined with non-spherical geometry of the object.

In deriving (11) and (12) we relied on the Newtonian limit of the Einstein's equations with Λ. This limit is the static version for the perturbation h_{00} ($g_{\mu H}=3_{\mu H}+h_{\mu H}$, $3_{\mu H}$ the Minkowski metric) of the linearized Einstein's equations ($h \equiv h^{\lambda}{}_{\lambda}$):

$$\partial^{\lambda}\partial_{\lambda}h_{\mu\nu} - \partial^{\lambda}\partial_{\mu}h_{\lambda\nu} - \partial^{\lambda}\partial_{\nu}h_{\lambda\mu} + \partial_{\mu}\partial_{\nu}h = -16\pi G_N S_{\mu\nu} - 2\Lambda\eta_{\mu\nu} \quad (14)$$

Besides the Newtonian limit which follows from it, is there any other effect in connection with (14)? The first what comes to mind are gravitational waves. Eq. (14) is like Fierz-Pauli equation for spin-2 with two sources: $S_{\mu H}$ and $\Lambda\eta_{\mu\nu}$. It goes without saying that (i) we cannot drop ad-hoc the Λ term in (14) while studying gravitational waves and (ii) the solution of (14) will contain the standard oscillatory terms and new terms due to Λ. Imposing the harmonic gauge condition (eq. (14) is gauge covariant under local gauge transformation ($h_{\mu H} \rightarrow h'_{\mu H} = h_{\mu H} + \partial_{\mu}e_H + \partial_H e_{\mu}$) and demanding the special solution of (14), which is proportional to Λ to be asymptotically to de Sitter metric (up to a diffeomorphism), we arrive at:

$$h_{\mu\nu} = \gamma_{\mu\nu} + \xi_{\mu\nu} \quad (15)$$

where $\Gamma_{\mu H}$ is the standard retarded solution (oscillatory for away from the source) and:

$$\xi_{00} = -\Lambda t^2 \quad ; \quad \xi_{0i} = \frac{2}{3}\Lambda t x_i \quad ; \quad \xi_{ij} = \Lambda t^2 \delta_{ij} + \frac{1}{3}\Lambda \varepsilon_{ij} \quad (16)$$

With $e_{ij} = x_i x_j$ for $i \neq j$ and 0 otherwise. The solution is in agreement with [12] when we let m→0. No observable can be calculated without the full solution i.e without the inclusion of $o_{\mu H}$. For instance, inserting $h_{\mu H}$ into the gravitational energy-momentum pseudo-tensor $t_{\mu H}$ and calculating the averaged components $<t_{oi}>$, (the averaged gravitational Poynting vector entering the expression for the power), we can infer when the part of $<t_{oi}>$ proportional to Λ becomes comparable to the standard oscillatory part for the case of monochromatic sources[4], this happens if the wave is produced at distances larger than:

$$O(f \hat{h} r_{\Lambda}^2) \quad (17)$$

Where f is the frequency of the incoming wave and $\hat{h}$ is the amplitude of the oscillatory part (wave) as arriving in the detector. Typical values of $\hat{h}$ are 10^{-20}-10^{-23} which converts $f \hat{h} r_{\Lambda}^2$ into an astrophysical distance in spite of the cosmological suppression factor r_{Λ}^2.

3. Two scales of gravity: G_N and Λ.

The cosmological constant Λ makes the gravity theory a two scales theory. There is a dual interplay between G_N and Λ which emerges in upper and lower bounds on some observables or in range of validity of certain approximations. For instance, in order for the Newtonian limit to be valid, one has to impose [7]:

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[4] If the source in non-monochromatic but a superposition of plane waves, the equation is still valid for each wavelength separately, then it is only necessary to analyze the most relevant wavelength, that which has enough amplitude to be detected and which has also enough frequency to reach earth.

$$R_{min} = r_s << r << R_{max} = \sqrt{6}r_\Lambda \quad ; \quad M << M_\Lambda \approx \frac{1}{G_N\sqrt{\Lambda}} \quad (18)$$

In the hydrostatic equilibrium to ensure global solution, the so-called Buchdald inequalities [13] require:

$$3r_S \le \frac{2}{3}R + R\sqrt{\frac{4}{9} - \frac{1}{3}\frac{R^2}{r_\Lambda^2}} \qquad (19)$$

This implies:

$$R \le \sqrt{\frac{4}{3}}r_\Lambda \text{ and } M \le \frac{2}{5}\sqrt{\frac{4}{9}}M_\Lambda \qquad (20)$$

Equation (20) is valid up to numerical factors of order one; the scales r_Λ and M_Λ enter again into these restrictions.

Another example is U_{eff} (See eq. (8)), from the equation of motion (7) in the de Sitter metric. To avoid that the first local maximum and the local minimum coincide to form a saddle point, one has to respect:

$$r_l > r_l^{min} = 2\sqrt{3}r_s \qquad (21)$$

If we use this result in the expression for elliptical orbits for elliptical orbits for non-relativistic (celestial) mechanics, we obtain:

$$R_{orbit} = \frac{r_l^2}{r_s} \to R_{orbit}^{max} \approx 0{,}55\, r_{max} \qquad (22)$$

which is now a non-relativistic restrictions on the size of possible bound orbits with non-zero angular momentum.

Since $\Lambda > 0$ allows a second local maximum beyond the minimum, we can repeat the procedure for this pair of extreme. The result reads [9]:

$$r_l < r_l^{max} = \left(\frac{3}{4}\right)^{1/3}\sqrt[3]{r_s^2 r_\Lambda} \qquad (23)$$

Again, the cosmological constant results into a new bound. Can we see this kind of duality also in other places? Yes, through another unexpected connection. In 1966 Andrei Sakharov proved in a general context that the maximal temperature of a black body radiation can not exceed [14]:

$$T < T_{max} \approx m_{pl} = \frac{1}{\sqrt{G_N}} \qquad (24)$$

This result seems too extra-orbitant large, $T_{max} \sim 10^{32}$K, to be of any use…, until combined with Hawking's formula for black hole evaporation:

$$T = \frac{1}{8\pi G_N M} \qquad (25)$$

Clearly a maximal temperature means here a minimal mass M_{min}:

$$M_{\min} = \frac{m_{pl}}{8\pi} \qquad (26)$$

Which means that Sakharov's result implies a black hole remnant of the mass $M_{\min}$ this is consistent with results obtained from Generalized Uncertainty Principle where the mass of the black hole remnant is of the order m_{pl} [15]. How can we invoke Λ along similar lines? Using $g_{00} = 1 - \frac{2r_s}{R} - \frac{R^2}{3r_\Lambda^2}$ from the Schwarzschild- de Sitter metric and demanding $g_{00} > 0$ to interpret $\int \sqrt{g_{00}}\, dx^0$ as proper time and trading the mass M in r_s for the constant density c for which we can use the Stefan-Boltzmann law $c = yT^4$, we finally obtain [16]:

$$0 < \rho < \frac{3}{8\pi}\frac{m_{Pl}^2}{R^2} - \frac{1}{8\pi}m_{Pl}^2 m_\Lambda^2 \qquad (27)$$

Obviously, we have to have $\frac{3}{R^2} > m_\Lambda^2$, which together with $R > \frac{1}{T}$ [17] gives:

$$T > T_{\min} = \frac{m_\Lambda}{\sqrt{3}} \qquad (28)$$

As before, we can use this in conjunction with Hawking's formula. Now, $T_{\max}$ defines a maximally possible mass:

$$M_{\max} = \frac{1}{8\pi}\frac{m_{Pl}^2}{m_\Lambda} = \frac{1}{8\pi G_N \sqrt{\Lambda}} = \frac{1}{8\pi}M_\Lambda \qquad (29)$$

One can check that the modifications which the Hawking's formula will receive due to Λ, will not substantially change (29).

Whereas $M_{\min}$ in (26) defines a minimum mass of a black hole, $M_{\max}$ in (29) apparently defines the maximum mass. From equations (26) and (29), (24) and (28) one can see the dual nature of G_N and Λ.

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