On optical characteristics of non-spherical scatterers

Abstract. In these brief notes we consider expressions defining some basic optical characteristics of non-spherical scatterers and point out the differences between these expressions and those for a more simple case of spherical particles.

1. Introduction

Electromagnetic radiation outside and inside any macroscopic body can be described by the vectors $\vec{E}, \vec{H}$ being the strengths of the electric and magnetic fields, respectively.

Usually one assumes that the radiation from a very distant source is incident on a scatterer and then this radiation can be represented by a plane wave

$$\vec{E}^{\rm in} = (E_\theta^{\rm in}\ \vec{i}_\theta + E_\varphi^{\rm in}\ \vec{i}_\varphi)\ e^{ik\vec{n}^{\rm in}\vec{r}}, \tag{1}$$

where $k = 2\pi/\lambda$ is the wavenumber, λ the radiation wavelength in the medium, the vector $\vec{n}^{\rm in}$ shows the direction of propagation of the wave, i is the complex unity, $\vec{r}$ the radius-vector. Here we introduce the spherical coordinate system (r, θ, φ), whose origin coincides with the origin of a Cartesian coordinate system that will be used below, $\vec{i}_\theta, \vec{i}_\varphi$ are the unit vectors.

Any plane wave can be represented by a superposition of two totally polarized waves – the TE and TM modes. For the former, the vector of the electric field is parallel to a selected plane, for the latter, it is perpendicular.

Besides that, one can consider only time-harmonic fields since any field can be expended in Fourier series whose components can be treated independently (see [1] for more details).

2. Scattering and Maxwell's equations

Scattering of electromagnetic radiation by an isolated macroscopic body is fully described by Maxwell's equations with the corresponding boundary conditions. In the general case

$$\nabla \cdot \vec{D} = \rho, \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{2}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \quad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}, \tag{3}$$

where D, B, ρ, J, M, t are the electric and magnetic inductions, the density of free changes, their current density, magnetization, and time, ε_0, μ_0 the dielectric permittivity and magnetic permeability of the medium surrounding the scatterer (for vacuum $\varepsilon_0 = \mu_0 = 1$). One can also introduce the material equations

$$\vec{J} = \sigma \vec{E}, \quad \vec{B} = \mu \vec{H}, \quad \vec{P} = \varepsilon_0 \chi \vec{E}, \tag{4}$$

where σ, χ are the conductivity and electric susceptibility.

The boundary conditions usually look as follows:

$$(\vec{E}^{\rm sca} + \vec{E}^{\rm in}) \times \vec{n} = \vec{E}^{\rm int} \times \vec{n}, \quad (\vec{H}^{\rm sca} + \vec{H}^{\rm in}) \times \vec{n} = \vec{H}^{\rm int} \times \vec{n}, \tag{5}$$

where all points on the scatterer surface S are considered, $\vec{n}$ is the external normal to S, $\vec{E}^{\rm in}, \vec{H}^{\rm in}$ mean the incident radiation, $\vec{E}^{\rm int}, \vec{H}^{\rm int}$ the field inside the scatterer, $\vec{E}^{\rm sca}, \vec{H}^{\rm sca}$ the additional (scattered) field appearing outside the scatterer.

For time-harmonic fields, from Eqs. (2)–(4) one can easily obtain the wave equation. For a spherically symmetric scatterer and a proper choice of the spherical coordinate system one can separate the variables both in the equation and the boundary conditions and obtain the solution in the explicit form (so called Mie theory – see, e.g., [1]). For a non-spherical scatterer of finite size such a separation is impossible in any coordinate system and solution of the scattering problem becomes more complicated (see, e.g., [2]). Here in contrast with the case of spheres, the proper choice of the directions of the axes of the Cartesian coordinate systems (there are usually two required systems – the laboratory one and that connected with the scatterer) plays in particular important role.

3. Amplitude matrix

If one considers the scattered field only in the far zone ($kr \gg 1$), it can be represented by a spherical wave (see [1] for more details)

$$\vec{E}^{\rm sca} = \left(E^{\rm sca}_{\theta,0}\ \vec{i}_\theta + E^{\rm sca}_{\varphi,0}\ \vec{i}_\varphi \right) \frac{e^{ikr}}{r}, \tag{6}$$

where $E^{\rm sca}_{\theta,0}, E^{\rm sca}_{\varphi,0}$ depend only on the direction of propagation of the scattered radiation.

For a scatterer of any shape, the linearity of Maxwell's equations leads to a linear dependence of the fields: the incident and internal ones, and the incident and scattered ones. In the second case, the relationship can be written using the amplitude matrix $\hat{S}$

$$\begin{pmatrix} E^{\rm sca}_\theta \\ E^{\rm sca}_\varphi \end{pmatrix} = \frac{e^{ikr}}{r} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} E^{\rm in}_\theta \\ E^{\rm in}_\varphi \end{pmatrix}. \tag{7}$$

Here to come to the standard form of Eq. (7), we include the known radial dependence of the scattered field in the components $E^{\rm sca}_\theta = E^{\rm sca}_{\theta,0}\ e^{ikr}/r$ and so on.

The elements of the amplitude matrix $\hat{S}$ on the directions of propagation of the incident and scattered radiations, the size, morphology and chemical composition of the scatterer as well as its orientation in the laboratory system.

In the coordinate system connected with the particle, the choice of the TE or TM mode (respectively, S_{11}, S_{21} or S_{12}, S_{22} are utilized) leads to two solutions for the scattered radiation $\vec{E}^{\rm sca}_{\rm TE}$ and $\vec{E}^{\rm sca}_{\rm TM}$.

4. Scattering matrix

Another way to describe the electromagnetic fields is to use the Stokes vector

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}, \tag{8}$$

connected with the field strengths as follows:

$$I = E_\theta E^*_\theta + E_\varphi E^*_\varphi, \quad Q = E_\theta E^*_\theta - E_\varphi E^*_\varphi, \quad U = -E_\theta E^*_\varphi - E_\varphi E^*_\theta, \quad V = i(E_\varphi E^*_\theta - E_\theta E^*_\varphi). \tag{9}$$

The process of scattering described above by the amplitude matrix can be also rewritten using the Stokes vectors

$$\vec{I}^{\rm sca} = \frac{1}{r^2}\ \hat{Z}\ \vec{I}^{\rm in}, \tag{10}$$

where the elements of the scattering matrix $\hat{Z}$ can be explicitly expressed via the elements of the amplitude matrix $\hat{S}$ (see [1] for more details).

For spheres, 10 out of 16 elements of the scattering matrix are equal to zero, whereas for non-spherical scatterers it is generally not the case, which can be important in applications (see, e.g., [3]).

5. Cross-sections and albedo

There are several characteristics of scattered radiation considered in the far zone which are suitable and often used in applications. For example, the extinction cross-section $C_{\rm ext}$ shows what fraction of the incident flux is absorbed and scattered by a particle. The product of the incident energy flux and this cross-section gives the power removed from the incident beam as a result of its interaction with the scatterer. This cross-section can be defined as follows:

$$C_{\rm ext} = \frac{4\pi}{k} {\rm Im} \left[\vec{E}^{\rm sca}, \vec{i} \right]_{\Theta=0}, \tag{11}$$

where Θ is the scattering angle, i.e. the angle between the directions of the incident and scattered radiation, $\vec{i}$ the unit vector parallel to $\vec{E}^{\rm in}$.

In the case of polarized incident radiation (see, e.g., [4])

$$C_{\rm ext} = \frac{1}{2}(C_{\rm ext}^{\rm TM} + C_{\rm ext}^{\rm TE}) + \frac{1}{2}(C_{\rm ext}^{\rm TM} - C_{\rm ext}^{\rm TE}) \cos 2\Psi \frac{Q^{\rm in}}{I^{\rm in}} + \frac{1}{2}(C_{\rm ext}^{\rm TM} - C_{\rm ext}^{\rm TE}) \sin 2\Psi \frac{U^{\rm in}}{I^{\rm in}}\,. \tag{12}$$

Here $(I^{\rm in}, Q^{\rm in}, U^{\rm in}, V^{\rm in})^{\rm T}$ is the Stokes vector of the incident radiation $\vec{I}^{\rm in}$, Ψ the positional angle of its polarization.

For spheres, obviously

$$C_{\rm ext} = \frac{1}{2}(C_{\rm ext}^{\rm TM} + C_{\rm ext}^{\rm TE}), \tag{13}$$

and $C_{\rm ext}^{\rm TM} = C_{\rm ext}^{\rm TE}$. For the scatterer of an arbitrary shape but non-polarized incident radiation, the expression (13) is also true.

The scattering cross-section shows what fraction of radiation is scattered by a particle in all directions (the product of the incident energy flux and this cross-section is equal to the total power of scattered radiation)

$$C_{\rm sca} = \frac{1}{I^{\rm in}} \int_{4\pi} I^{\rm sca} d\Omega, \tag{14}$$

where $I^{\rm in}, I^{\rm sca} = |\vec{E}^{\rm sca}|^2$ are the intensities of the incident and scattered radiation, respectively.

From the energy conservation law, it follows that the absorption cross-section is (see [1] for more details)

$$C_{\rm abs} = C_{\rm ext} - C_{\rm sca}. \tag{15}$$

The important parameter is the single scattering albedo Λ defined as the fraction of the scattered radiation in the total radiation excluded from the incident beam

$$\Lambda = \frac{C_{\rm sca}}{C_{\rm ext}}, \tag{16}$$

for non-polarized incident radiation

$$\Lambda = \frac{C_{\rm sca}^{\rm TM} + C_{\rm sca}^{\rm TE}}{C_{\rm ext}^{\rm TM} + C_{\rm ext}^{\rm TE}}. \tag{17}$$

6. Asymmetry factor and radiation pressure

The intensity of radiation scattered in different directions is proportional to the phase function. For spheres, the phase function has the rotational symmetry relative to the direction of propagation of the incident radiation. For non-spherical particles, such a symmetry is generally absent. Therefore, while for spheres the phase function asymmetry factor g (or $\langle \cos \Theta \rangle$) is a scalar, for non-spherical particles it is a vector with the components (see also [5])

$$g_{\rm x,y,z} = \frac{1}{C_{\rm sca}} \int_{4\pi} \vec{n}^{\rm sca}\ \vec{i}_{\rm x,y,z}\ I^{\rm sca}\ d\Omega, \tag{18}$$

where $\vec{n}^{\rm sca}$ is the direction of propagation of the scattered radiation, $\vec{i}_{\rm x,y,z}$ are the unit vectors of a Cartesian system.

The absence of symmetry in the spatial distribution of the scattered radiation leads to the fact that the total recoil of scattered photons is not directed along the direction of propagation of the incident radiation. Therefore, in contrast to the spheres, for non-spherical particles the cross-section of radiation pressure is a vector too

$$\vec{C}_{\rm pr} = C_{\rm ext} \frac{\vec{R}}{R} - C_{\rm sca}\vec{g}, \tag{19}$$

where R is the distance between the scatterer and the radiation source. For spheres,

$$C_{\rm pr} = C_{\rm ext} - gC_{\rm sca} = C_{\rm abs} + (1-g)C_{\rm sca}. \tag{20}$$

7. Absorption and re-emission

The absorption process can be described as follows:

$$\vec{I}^{\rm after} = \hat{R}^{-1}\,\hat{\Lambda}_{\rm part}\,\hat{R}\,\vec{I}^{\rm before}\,, \tag{21}$$

where $\vec{I}^{\rm before}$ ($\vec{I}^{\rm after}$) is the Stokes vector of radiation before (after) absorption.

Here the matrix $\hat{R}$

$$\hat{R} = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \cos 2\Psi & \sin 2\Psi & 0 \\ 0 & -\sin 2\Psi & \cos 2\Psi & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{22}$$

makes the transition from the laboratory system to the system connected with the scatterer, and $\hat{\Lambda}_{\rm part}$ is the albedo matrix in the particle system

$$\hat{\Lambda}_{\rm part} = \left(\begin{array}{cccc} \tilde{l}_1 & \tilde{l}_2 & 0 & 0 \\ \tilde{l}_2 & \tilde{l}_1 & 0 & 0 \\ 0 & 0 & \tilde{l}_1 & 0 \\ 0 & 0 & 0 & \tilde{l}_1 \end{array}\right)\,. \tag{23}$$

Here $\tilde{l}_1$ and $\tilde{l}_2$ are

$$\begin{aligned} \tilde{l}_1 &= \frac{\Lambda^{\rm TM}}{1 + C^{\rm TM}_{\rm ext}/C^{\rm TE}_{\rm ext}} + \frac{\Lambda^{\rm TE}}{1 + C^{\rm TE}_{\rm ext}/C^{\rm TM}_{\rm ext}}, \\ \tilde{l}_2 &= \frac{\Lambda^{\rm TM}}{1 + C^{\rm TM}_{\rm ext}/C^{\rm TE}_{\rm ext}} - \frac{\Lambda^{\rm TE}}{1 + C^{\rm TE}_{\rm ext}/C^{\rm TM}_{\rm ext}}\,, \end{aligned} \tag{24}$$

where $\Lambda^{\rm TE,TM} = C^{\rm TE,TM}_{\rm sca}/C^{\rm TE,TM}_{\rm ext}$ is the albedo of totally polarized radiation. In the case of spherical particles instead of $\hat{\Lambda}$ one has multiplication by a number – the value of albedo.

In the case of re-emission of radiation, the Stokes vector can be written in the particle system as follows:

$$\vec{I}^{\rm emis} \propto \left(\begin{array}{c} (C^{\rm TM}_{\rm abs} + C^{\rm TE}_{\rm abs})/2 \\ -(C^{\rm TM}_{\rm abs} - C^{\rm TE}_{\rm abs})/2 \\ 0 \\ 0 \end{array}\right) B_\lambda(T_{\rm d}), \tag{25}$$

where $B_\lambda(T_{\rm d})$ is the Plank function, $T_{\rm d}$ the temperature of dust grains.

References

[1] Bohren C.F., Huffman D.R. (1983) Absorption and Scattering of Light by Small Particles. J.Wiley & Sons, New York.

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[5] Draine B.T., Flatau (1997) Princeton Obs. Preprint POPe-695.